Trig 29.03 Paper I *[198 marks]*

1. By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \,\mathrm{d}x$ in terms of n, where n is a non-zero real number. *[6 marks]*

2a. Show that $2x - 3 - \frac{6}{x-1} = \frac{2x^2-5x-3}{x-1},\ x \in \mathbb{R},\ x \neq 1.$ *[2 marks]*

2b. Hence or otherwise, solve the equation $2 \sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi,\ \theta \neq \frac{\pi}{4}$. *[5 marks]*

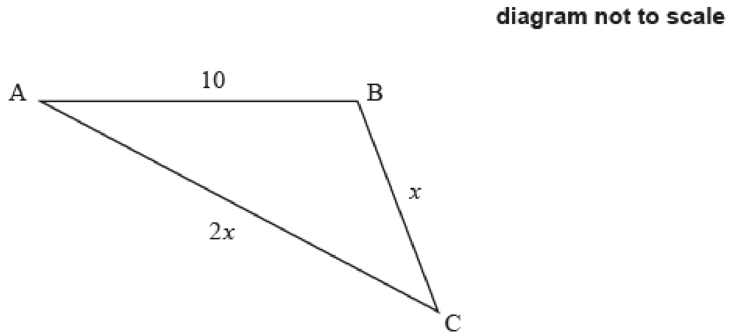
3a. Show that $\sin 2x + \cos 2x - 1 = 2 \sin x(\cos x - \sin x)$. *[2 marks]*

3b. Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. *[6 marks]*

4a. Show that the equation $2 \cos^2 x + 5 \sin x = 4$ may be written in the form $2 \sin^2 x - 5 \sin x + 2 = 0$. *[1 mark]*

4b. Hence, solve the equation $2 \cos^2 x + 5 \sin x = 4, 0 \leq x \leq 2\pi$. *[5 marks]*

5. The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$. *[7 marks]*

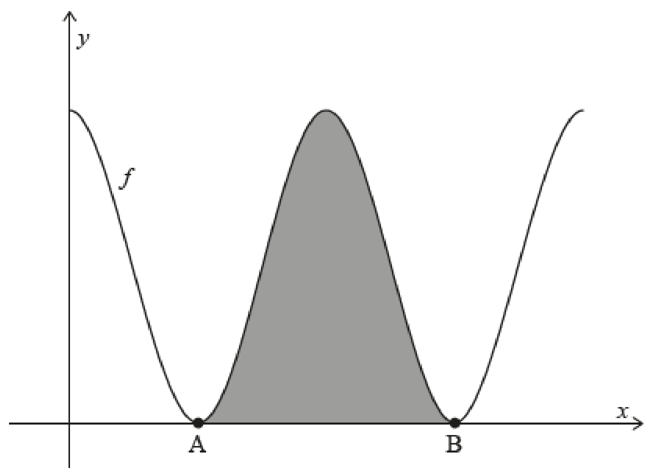


Given that $\cos \widehat{\mathrm{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^{+}$.

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.



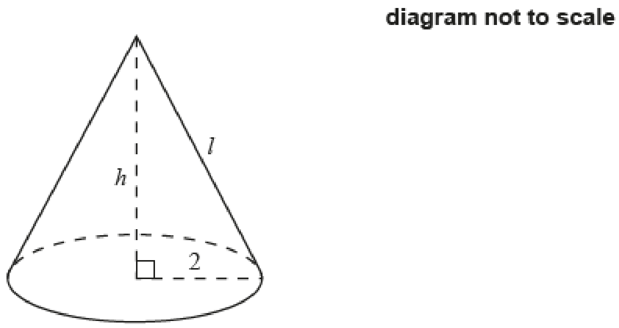
The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

6a. Find the x-coordinates of A and B. *[3 marks]*

6b. Show that the area of the shaded region is 12π. *[5 marks]*

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

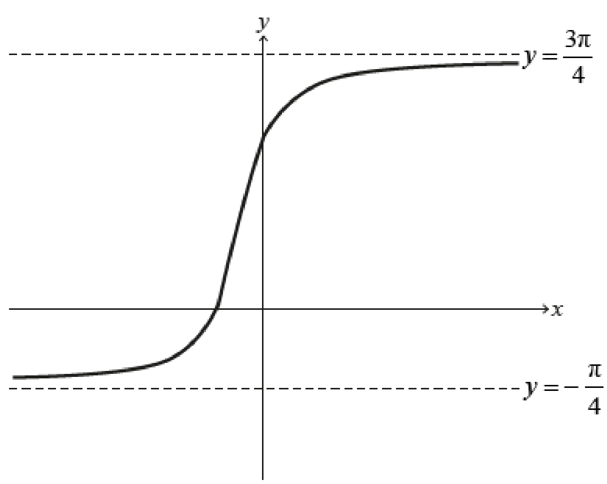


6c. Find the value of l. *[3 marks]*

6d. Hence, find the volume of the cone. *[4 marks]*

7. It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. *[4 marks]*

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



8a. Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. *[3 marks]*

8b. Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$. *[4 marks]*

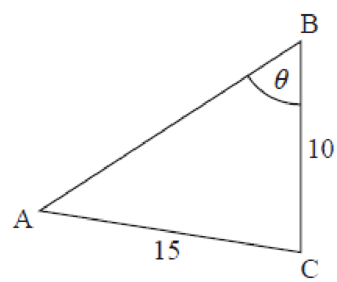
8c. Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}, x > 0$. *[3 marks]*

8d. Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. *[9 marks]*

9. Solve the equation $2\cos^2 x + 5\sin x = 4, 0 \leq x \leq 2\pi$. *[7 marks]*

The following diagram shows a triangle ABC.

diagram not to scale



$\mathrm{AC} = 15\text{ cm}, \mathrm{BC} = 10\text{ cm}$, and $\mathrm{A\hat{B}C} = \theta$.

Let $\sin \mathrm{C\hat{A}B} = \frac{\sqrt{3}}{3}$.

10a. Given that $\mathrm{A\hat{B}C}$ is acute, find $\sin\theta$. *[3 marks]*

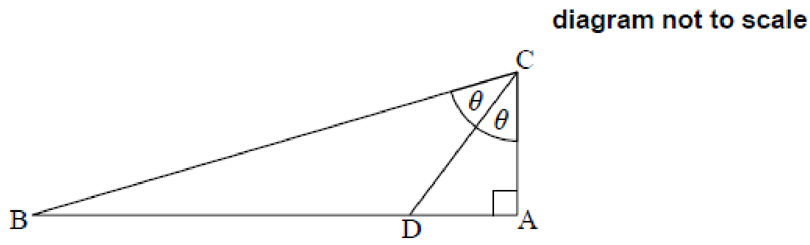
10b. Find $\cos\left(2 \times \mathrm{C\hat{A}B}\right)$. *[3 marks]*

11. Let $f(x) = 4\cos\left(\frac{x}{2}\right) + 1$, for $0 \leqslant x \leqslant 6\pi$. Find the values of x for which $f(x) > 2\sqrt{2} + 1$. *[8 marks]*

12. A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$. *[7 marks]*

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.

The following diagram shows a right triangle ABC. Point D lies on AB such that CD bisects $\hat{ACB}$.



$\hat{ACD} = \theta$ and AC = 14 cm

13a. Given that $\sin\theta = \frac{3}{5}$, find the value of $\cos\theta$. *[3 marks]*

13b. Find the value of $\cos 2\theta$. *[3 marks]*

13c. Hence or otherwise, find BC. *[2 marks]*

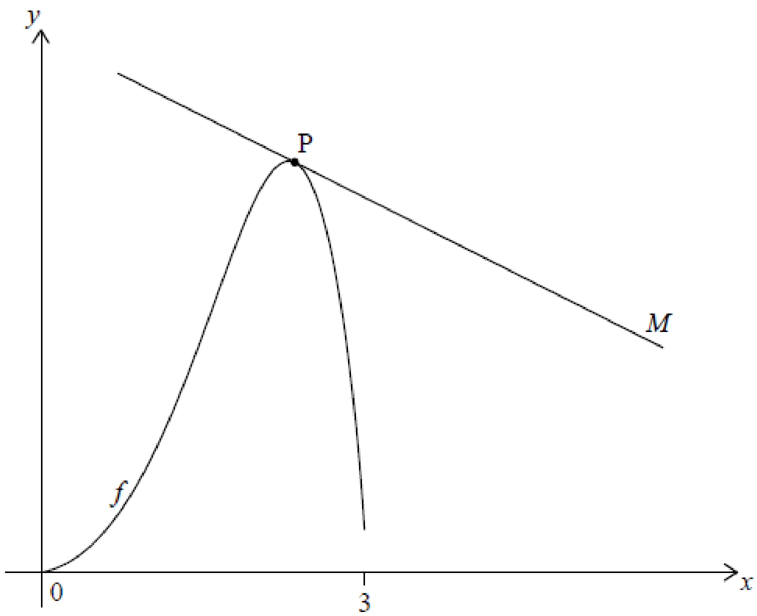
Let θ be an **obtuse** angle such that $\sin\theta = \frac{3}{5}$.

14a. Find the value of $\tan\theta$. *[4 marks]*

14b. Line L passes through the origin and has a gradient of $\tan\theta$. Find the equation of L. *[2 marks]*

Let $f(x) = \mathrm{e}^x \sin x - \frac{3x}{4}$.

14c. The following diagram shows the graph of f for $0 \le x \le 3$. Line M is a tangent to the graph of f at point P. *[4 marks]*



Given that M is parallel to L, find the x-coordinate of P.

The lengths of two of the sides in a triangle are 4 cm and 5 cm. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}$ cm^2.

15a. Show that $\sin\theta = \frac{\sqrt{15}}{4}$. *[1 mark]*

15b. Find the two possible values for the length of the third side. *[6 marks]*

16a. Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$. *[2 marks]*

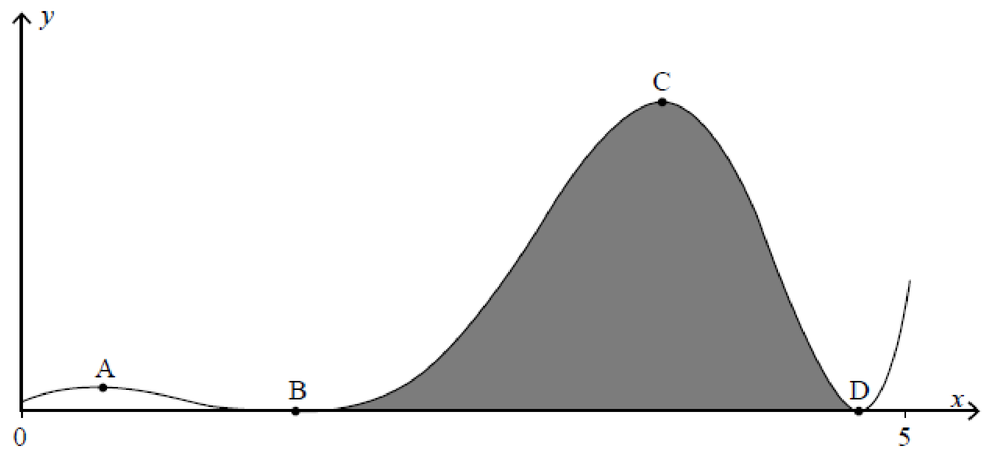
16b. Show that $\sec 2x + \tan 2x = \frac{\cos x + \sin x}{\cos x - \sin x}$. *[4 marks]*

16c. Hence or otherwise find $\int_0^{\frac{\pi}{6}} (\sec 2x + \tan 2x)\,\mathrm{d}x$ in the form $\ln\left(a + \sqrt{b}\right)$ where $a, b \in \mathbb{Z}$. *[9 marks]*

17a. Use integration by parts to show that $\int \mathrm{e}^x \cos 2x \mathrm{d}x = \frac{2\mathrm{e}^x}{5}\sin 2x + \frac{\mathrm{e}^x}{5}\cos 2x + c, \ c \in \mathbb{R}$. *[5 marks]*

17b. Hence, show that $\int \mathrm{e}^x \cos^2 x \mathrm{d}x = \frac{\mathrm{e}^x}{5}\sin 2x + \frac{\mathrm{e}^x}{10}\cos 2x + \frac{\mathrm{e}^x}{2} + c$, $c \in \mathbb{R}$. *[3 marks]*

The function f is defined by $f(x) = \mathrm{e}^x \cos^2 x$, where $0 \le x \le 5$. The curve $y = f(x)$ is shown on the following graph which has local maximum points at A and C and touches the x-axis at B and D.



17c. Find the x-coordinates of A and of C , giving your answers in the form $a + \arctan b$, where a, $b \in \mathbb{R}$. *[6 marks]*

17d. Find the area enclosed by the curve and the x-axis between B and D, as shaded on the diagram. *[5 marks]*

18a. Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where r, $\theta \in \mathbb{R}^+$. *[5 marks]*

Let *S* be the sum of the roots found in part (a).

18b. Show that Re *S* = Im *S*. *[4 marks]*

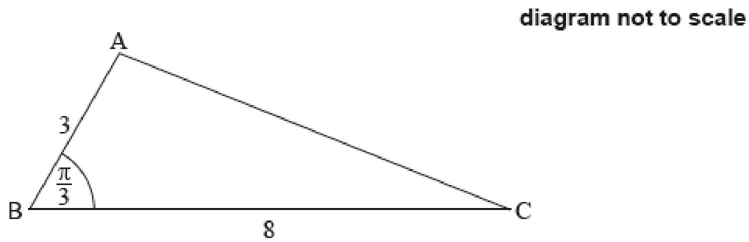
18c. By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of $\cos\frac{\pi}{12}$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are integers to be determined. *[3 marks]*

18d. Hence, or otherwise, show that $S = \frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$. *[4 marks]*

19. Let $a = \sin b,\ 0 < b < \frac{\pi}{2}$. *[5 marks]*

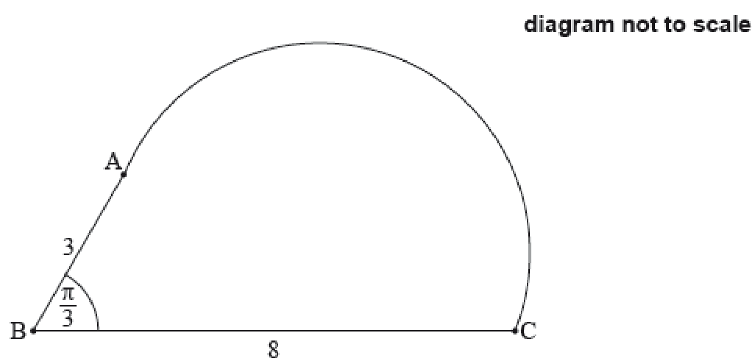
Find, in terms of *b*, the solutions of $\sin 2x = -a,\ 0 \leqslant x \leqslant \pi$.

The following diagram shows triangle ABC, with $\mathrm{AB} = 3\mathrm{cm}$, $\mathrm{BC} = 8\mathrm{cm}$, and $\mathrm{A\hat{B}C} = \frac{\pi}{3}$.



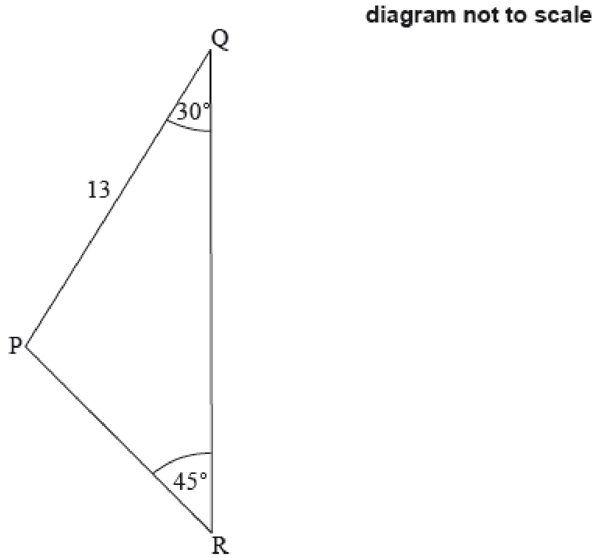
20a. Show that $\mathrm{AC} = 7\ \mathrm{cm}$. *[4 marks]*

20b. The shape in the following diagram is formed by adding a semicircle with diameter [AC] to the triangle. *[3 marks]*



Find the exact perimeter of this shape.

21. The following diagram shows triangle PQR. *[6 marks]*



$\text{P}\hat{\text{Q}}\text{R} = 30°$, $\text{Q}\hat{\text{R}}\text{P} = 45°$ and $\text{PQ} = 13\text{ cm}$.

Find PR.

22. Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$. *[5 marks]*