For questions 7 to 12, use the method demonstrated in Worked Example 10.2 to find the values of a and b which make the following functions continuous and differentiable at all points.

7 **a** $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ ax + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^3 & x \leqslant 1 \\ ax + b & x > 1 \end{cases}$

8 **a** $f(x) = \begin{cases} x^2 + a & x \leqslant 2 \\ bx + 1 & x > 2 \end{cases}$

b $f(x) = \begin{cases} x^2 + a & x \leqslant 3 \\ bx + 2 & x > 3 \end{cases}$

9 **a** $f(x) = \begin{cases} x^a & x \leqslant 1 \\ bx - 1 & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^a & x \leqslant 1 \\ bx - 2 & x > 1 \end{cases}$

10 **a** $f(x) = \begin{cases} e^x & x \leqslant 0 \\ ax + b & x > 0 \end{cases}$

b $f(x) = \begin{cases} e^x & x \leqslant 2 \\ ax + b & x > 2 \end{cases}$

11 **a** $f(x) = \begin{cases} e^{-x} & x \leqslant 1 \\ \frac{a}{x} + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} e^{-x} & x \leqslant 2 \\ \frac{a}{x} + b & x > 2 \end{cases}$

12 **a** $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ a\ln x + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ a\sqrt{x} + b & x > 1 \end{cases}$

9 Use L'Hôpital's rule to find $\lim_{x\to 0} \frac{e^x - 1}{x}$.

10 Use L'Hôpital's rule twice to find $\lim_{x\to\infty} \frac{x^2}{e^x}$.

11 Find $\lim_{x\to 0} \frac{1 - \cos(x^2)}{x^4}$.

12 Find the following limits.

a $\lim_{x\to 0} \frac{x - \cos x}{x + \cos x}$ **b** $\lim_{x\to 0} \frac{x - \sin x}{x + \sin x}$

13 Find the following limits.

a $\lim_{x\to 0} \frac{\sin x}{x^2}$ **b** $\lim_{x\to 2} \frac{\sin(x-2)}{x^2 - 4}$

14 Evaluate $\lim_{x\to\frac{\pi}{2}} \frac{\cos^2(5x)}{\cos^2 x}$.

15 Evaluate $\lim_{x\to 1} \frac{(\ln x)^2}{x^2 - 2x + 1}$.

16 Evaluate $\lim_{x\to 1} \frac{(\ln x)^2}{x^3 + x^2 - 5x + 3}$.

18 Show that the function

$$f(x) = \begin{cases} \frac{\sin 3x}{\sin x} & x \leqslant 0 \\ \frac{e^{3x} - 1}{x} & x > 0 \end{cases}$$

is continuous at $x = 0$.

19 Find $\lim_{x\to -\infty} xe^x$.

20 Find $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

21 Find $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$.

22 Use L'Hôpital's rule to prove that

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = 1.$$

12 One side of a rectangle lies on the x-axis and two corners lie on the curve $y = \sin x$, $0 \leqslant x \leqslant \pi$. Find the largest possible area of the rectangle.

13 Find the closest distance from the point (1, 2) to the curve $y = x^3$.

14 A piece of wire is bent to form an isosceles triangle. Prove that the largest possible area is formed when the triangle is equilateral.

15 Two corridors meet at a right angle. Find the longest ladder that would fit horizontally around the corner if

a both corridors are 1 m wide

b one corridor is 1 m wide and the other corridor is 8 m wide.

You may assume for your calculations that the ladder has negligible width.

14 Find the tangent to the curve $\frac{x+y}{x-y} = 2y$ at the point (3, 1).

15 A curve has equation $\sin(x + y) = \sqrt{2}\cos(x - y)$.

a Show that the point $\left(\frac{13\pi}{24}, \frac{5\pi}{24}\right)$ lies on the curve.

b Find the gradient of the curve at this point, giving your answer in the form $a + b\sqrt{3}$.

16 Find the equations of the two possible tangents to the curve $x^2 + 3xy + y^2 = 1$ when $x = 0$.

17 a Find the y intercepts of the curve $y^3 - y - x = 0$.

b Find the gradients of the tangents at each of these points.

18 a Find the x intercepts of the curve $e^y - y = x^2$.

b Find the equations of the tangents at each of these points.

19 a Find the possible values of y with an x coordinate of 1 in the equation $x^2 - 5xy + y^2 = 1$.

b Find the equations of the tangents at each of these points.

20 Find $\frac{dy}{dx}$ if $e^y - x\sin y = \ln y$.

21 Given that $x\sin x = y\sin y$, find an expression for $\frac{dy}{dx}$ in terms of x and y.

22 If $x^2 + y^2 = 9$, find an expression for $\frac{d^2y}{dx^2}$ in terms of y.

23 Find the coordinates of the stationary points on the curve $x^2 + 4xy + 2y^2 + 1 = 0$.

24 Find the coordinates of the turning points on the curve $y^3 - 3xy^2 + x^3 = 8$.

25 a Sketch the curve $y^2 = x^3$.

b Show that the equation of the tangent to the curve at the point (4, 8) is $y = 3x - 4$.

c Find the coordinates of the point where this tangent meets the curve again.

11 An x by y rectangle is expanding with $\frac{dx}{dt} = 4\,\text{cm s}^{-1}$ and $\frac{dy}{dt} = -2\,\text{cm s}^{-1}$. When $x = 3$ cm and $y = 4$ cm, find

a the rate of increase of the rectangle's area

b the rate of increase of the length of the diagonal.

12 An inverted cone is being filled with water at a constant rate of $5\,\text{cm}^3\,\text{s}^{-1}$. The surface of the water is always horizontal as it is being filled. The largest diameter of the cone is 10 cm and its height is 30 cm. If the volume of water in the cone is V at time t, and h is the height of the water above the vertex of the cone,

a show that $V = \frac{\pi h^3}{108}$

b find the rate that the height is increasing when $h = 18$ cm.

13 A circular stain of radius r cm and area $A\,\text{cm}^2$ is increasing in size. At a certain time, the rate of increase of the radius is $1.8\,\text{cm s}^{-1}$ and the rate of increase of the area is $86.5\,\text{cm}^2\,\text{s}^{-1}$. Find the radius of the stain at this point.

14 A sportsman throws a ball. When it is 2 m above the sportsman and 4 m away horizontally it is moving with purely horizontally with a speed of $3\,\text{m s}^{-1}$. Find the rate at which the ball is moving away from the sportsman.

15 The density of a reactive substance is given by its mass divided by its volume. When the density is $5\,\text{g cm}^3$ the mass is decreasing at a rate of $2\,\text{g s}^{-1}$ and the volume is decreasing at a rate of $1\,\text{cm}^3\,\text{s}^{-1}$. Determine, with justification, whether the density is increasing or decreasing.

16 A ladder of length 3 m is sliding down a vertical wall. The foot of the ladder is on horizontal ground. When the point of contact with the wall is 2 m above the horizontal that point is moving down at a rate of $0.1\,\text{m s}^{-1}$. At what speed is the foot of the ladder moving away from the wall, assuming that the ladder always stays in contact with both the wall and the ground?