Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set A

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 30]*

This question is about investigating and proving properties of a sequence called the Fibonacci sequence.

The Fibonacci sequence is defined by the initial conditions $F_1 = 1, F_2 = 1$

and the recursion relation

$F_{n+2} = F_n + F_{n+1}$ for $n \geqslant 1$.

a Write down F_3, F_4 and F_5. *[3]*

b Dominique suggests that 1 is the only Fibonacci number which is a perfect square.
Use a counterexample to disprove this statement. *[2]*

c Prove by induction that

$$\sum_{i=1}^{i=n} (F_i)^2 = F_n F_{n+1}.$$ *[6]*

d Find the smallest value of k such that $F_k \geqslant k$. Prove that $F_n \geqslant n$ for $n \geqslant k$. *[7]*

e It is suggested that $F_n = \alpha^n$ might satisfy the recursion relation. Given that $\alpha \neq 0$, find the two possible values of α. *[4]*

f Show that if α_1 and α_2 are the two possible values of α then $F_n = A\alpha_1^n + B\alpha_2^n$, where A and B are constants, also satisfies the recursion relation. *[2]*

g Find an expression for F_n in terms of n. *[4]*

h Hence find the value of $\frac{F_{n+1}}{F_n}$ as n tends to infinity. *[2]*

2 *[Maximum mark: 25]*

This question is about resonance in vibrating objects.

a Write down the period of the function $\cos \pi t$. *[1]*

b **i** Sketch the function $y = \cos \pi t + \cos 2\pi t$ for $0 \leqslant t \leqslant 3$.

ii Write down the period of the function $\cos \pi t + \cos 2\pi t$. *[2]*

c **i** Use technology to investigate the period of the given functions below. Write down the values of A, B and C.

f(t)	Period
$\cos \pi t + \cos 1.5\pi t$	A
$\cos \pi t + \cos 1.25\pi t$	B
$\cos \pi t + \cos 1.1\pi t$	C

ii Hence conjecture an expression for the period, T, of $f(t) = \cos \pi t + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right)$ where n is an integer. *[4]*

d Prove that, for your conjectured value of T, $f(t + T) = f(t)$. *[3]*

e **i** Use the compound angle formula to write down and simplify an expression for $\cos(A + B) + \cos(A - B)$.

ii Hence find a factorized form for the expression $\cos P + \cos Q$. *[3]*

f By considering the factorized form of f(t), explain the shape of its graph. *[3]*

g A piano string oscillates when plucked. The displacement, x, from equilibrium as a function of time is modelled by:

$$\frac{d^2x}{dt^2} + 4x = 0.$$

Show that a function of the form $x = f(t) = \cos(\omega t)$ solves this differential equation for a positive value of ω to be stated. *[4]*

h The piano string can be subjected to an external driving force from a tuning fork. The differential equation becomes:

$$\frac{d^2x}{dt^2} + 4x = \cos kt.$$

Find a solution of the form $x = f(t) + g(k) \cos kt$ where $g(k)$ is a function to be found. *[3]*

i Resonance is a phenomenon in which the amplitude of the driven oscillation grows without limit. For what positive value of k will resonance occur? Justify your answer. *[2]*