# Mathematics: analysis and approaches
# Higher level
# Paper 3 Practice Set B

Candidate session number

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1 hour

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1** *[Maximum mark: 30]*

*This question is about using sums of sequences to investigate the formula for integrating a polynomial.*

**a** Prove by induction that

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$ [7]

**b** Simplify $(n + 1)^3 - n^3$. [2]

**c** By considering two different ways of expressing

$$\sum_{r+1}^{n} (r+1)^3 - r^3$$

show that

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.$$ [7]

**d** By considering splitting the region into $n$ rectangles each of width $\frac{x}{n}$ and whose top right corner lies on the curve $y = x^2$, show that

$$\int_0^x t^2 \, dt \leqslant \frac{x}{n} \sum_{r=1}^{n} \left(\frac{rx}{n}\right)^2.$$ [4]

**e** By considering rectangles whose top left corner lies on the curve $y = x^2$, form a similar inequality to provide a lower bound on $\int_0^x t^2 \, dt$. [4]

**f** By considering the limit as $n \to \infty$ prove that

$$\int_0^x t^2 \, dt = \frac{x^3}{3}.$$ [6]

**2** *[Maximum mark: 25]*

*This question is about estimating parameters from data.*

Let $X_1$ and $X_2$ both be random variables representing independent observations from a population with mean $\mu$ and variance $\sigma^2$.

You may use without proof in this question the fact that

$$\mathrm{E}(aX_1 + bX_2) = a\,\mathrm{E}(X_1) + b\,\mathrm{E}(X_2)$$

and

$$\mathrm{Var}(aX_1 + bX_2) = a\,\mathrm{Var}(X_1) + b\,\mathrm{Var}(X_2).$$

**a** Find an expression for $\bar{X}$, the random variable representing the sample mean of the two observed values. *[1]*

**b** Show that $\mathrm{E}(\bar{X}) = \mu$ and find an expression for $\mathrm{Var}(\bar{X})$ in terms of $\sigma$. *[4]*

The sample variance is defined as

$$S^2 = \frac{X_1^2 + X_2^2}{2} - \bar{X}^2$$

**c** **i** Find $\mathrm{E}(X^2)$ in terms of $\mathrm{Var}(X)$ and $\mathrm{E}(X)$.

**ii** Show that $\mathrm{E}(S^2) = \frac{1}{2}\sigma^2$. *[4]*

An unbiased estimator of a population parameter is one whose expected value equals the population parameter.

**d** **i** Show that $M = \frac{2X_1 + 3X_2}{5}$ is an unbiased estimator of $\mu$.

**ii** When comparing two unbiased estimators, the one with a lower variance is said to be more efficient. Determine whether $M$ or $\bar{X}$ is a more efficient unbiased estimator of $\mu$. *[5]*

In a promotion, tokens are placed at random in boxes of cereal. $Y$ is the random variable describing the number of boxes of cereal that need to be opened, up to and including the one where a token is found. Two independent investigations were conducted.

**e** The tokens are placed in cereal boxes with probability $p$. The presence of a token in a cereal box is independent of other boxes.

**i** Find an expression for $L$, the probability of observing $Y_1 = a$ and $Y_2 = b$ in terms of $a$, $b$ and $p$.

**ii** Find the value of $p$ which maximizes $L$. This is called the maximum likelihood estimator of $p$. *[8]*

In the first observation, $Y$ was found to be 4. In the second observation $Y$ was found to be 8.

**f** **i** Find an unbiased estimate for the variance of $Y$.

**ii** Find a maximum likelihood estimate for $p$. *[3]*