Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set C

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 25]*

This question is about investigating two sequences involving paired parentheses and using links between them to generate a formula.

In computer science, a useful validity check for mathematical expressions is to check that all open parentheses match closed parentheses.

A_n is the number of possible expressions (not necessarily correct) with n pairs of parentheses. For example, when $n = 1$ there are two possible expressions:

() or)(

So $A_1 = 2$.

a Show that $A_2 = 6$. *[2]*

b When $n = 8$:

i How many characters are in the expression?

ii How many of these positions should be chosen to contain open '(' parentheses?

iii Hence find A_8 and state a general expression for A_n. *[4]*

B_n is the number of correct expressions with n pairs of parentheses.

For example, when $n = 1$ the only correct expression is (), so $B_1 = 1$.

c **i** Show that $B_2 = 2$.

ii Find B_3. *[3]*

You are given that $B_8 = 1430$.

d There is a relationship between A_n and B_n of the form

$$B_n = \mathrm{f}(n)\, A_n.$$

Suggest an appropriate function for $\mathrm{f}(n)$ and hence suggest an expression for B_n. *[3]*

e It can be shown that $B_{n+1} = \dfrac{4n+2}{n+2} B_n$. Use this result to prove by induction your conjecture from part **d**. *[8]*

f A corrupted computer file includes an expression involving 20 pairs of brackets. The corruption results in the order of the brackets being switched at random.

What is the probability that the end result still has brackets which form a 'correct expression'? *[2]*

g Asher and Elsa are running in an election. 100 voters cast their vote one at a time, and a running total is kept. At the end of the voting the result is a tie with each candidate getting 50 votes.

What is the probability that Asher is never ahead of Elsa at any point during the count? Justify your answer. *[3]*

2 *[Maximum mark: 30]*

This question is about the path of three snails chasing after each other.

a Find $\left|\mathrm{e}^{\frac{2\mathrm{i}\pi}{3}} - 1\right|$. *[3]*

Three snails – Alf, Bill and Charlotte – are positioned on the vertices of an equilateral triangle whose centre of rotational symmetry is the origin of the Argand plane. Alf is positioned at the point $z = 1$ and Bill is above the real axis.

b Find the positions of the other two snails. *[2]*

c If Bill is stationary and Alf moves towards him at speed 1 unit per second, how far does Alf travel until he reaches Bill? How long does it take Alf to get there? *[2]*

Alf chases Bill, Bill chases Charlotte and Charlotte chases Alf. They all travel with speed 1 unit per second. The position of Alf at time t is denoted by z_A and the position of Bill is denoted by z_B.

d Explain why $\frac{dz_A}{dt} = \frac{z_B - z_A}{|z_B - z_A|}$. *[2]*

e Write z_B in terms of z_A. *[1]*

f If $z_A = re^{i\theta}$, find an expression for $\frac{dz_A}{dt}$ in terms of r, θ, $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[2]*

g Hence, by comparing real and imaginary parts, find differential equations for $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[7]*

h Solve these differential equations. *[7]*

i How long does it take Alf to reach Bill? How far has Alf travelled until he reaches Bill? How many rotations does he make around the origin? *[4]*