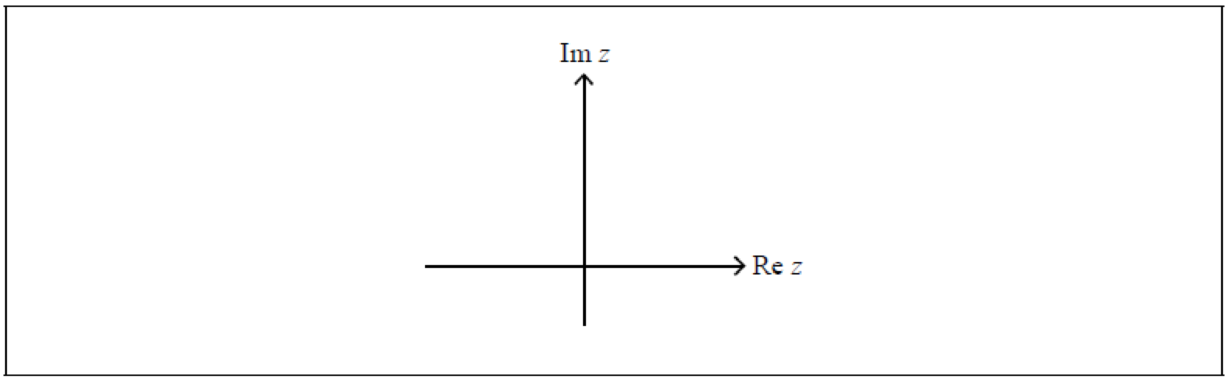
Complex numbers 21.04 *[56 marks]*

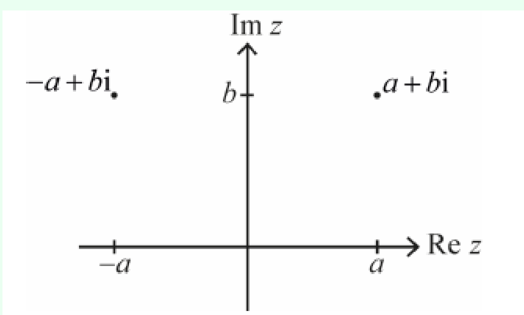
Let $z = a + b\mathrm{i}$, a, $\mathrm{b} \in \mathbb{R}^{+}$ and let $\arg z = \theta$.

1a. Show the points represented by z and $z - 2a$ on the following Argand diagram. *[1 mark]*



Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.



A1

Note: Award ***A1*** for z in first quadrant and $z - 2a$ its reflection in the y-axis.

[1 mark]

1b. Find an expression in terms of θ for $\arg\left(z - 2a\right)$. *[1 mark]*

Markscheme

$\pi - \theta$ (or any equivalent) ***A1***

[1 mark]

1c. Find an expression in terms of θ for $\arg\left(\frac{z}{z-2a}\right)$. *[2 marks]*

Markscheme

$\arg\left(\frac{z}{z-2a}\right) = \arg(z) - \arg(z-2a)$ ***(M1)***

$= 2\theta - \pi$ (or any equivalent) ***A1***

[2 marks]

1d. Hence or otherwise find the value of θ for which $\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0$. *[3 marks]*

Markscheme

METHOD 1

if $\text{Re}\left(\frac{z}{z-2a}\right)=0$ then $2\theta-\pi=\frac{n\pi}{2}$, ($n$ odd) ***(M1)***

$-\pi<2\theta-\pi<0\Rightarrow n=-1$

$2\theta-\pi=-\frac{\pi}{2}$ ***(A1)***

$\theta=\frac{\pi}{4}$ ***A1***

METHOD 2

$\frac{a+b\text{i}}{-a+b\text{i}}=\frac{b^2-a^2-2ab\text{i}}{a^2+b^2}$ ***M1***

$\text{Re}\left(\frac{z}{z-2a}\right)=0\Rightarrow b^2-a^2=0$

$b=a$ ***A1***

$\theta=\frac{\pi}{4}$ ***A1***

Note: Accept any equivalent, *eg* $\theta=-\frac{7\pi}{4}$.

[3 marks]

2a. Find the roots of the equation $w^3=8\text{i}$, $w\in\mathbb{C}$. Give your answers in Cartesian form. *[4 marks]*

Markscheme

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METHOD 1

$w^3 = 8\mathrm{i}$

writing $8\mathrm{i} = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find cube roots of w using modulus-argument form.

cube roots $w = 2\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)\right)$ ***(M1)***

i.e. $w = \sqrt{3} + \mathrm{i},\ -\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

METHOD 2

$w^3 + (2\mathrm{i})^3 = 0$

$(w + 2\mathrm{i})\left(w^2 - 2w\mathrm{i} - 4\right) = 0$ ***M1***

$w = \frac{2\mathrm{i} \pm \sqrt{12}}{2}$ ***M1***

$w = \sqrt{3} + \mathrm{i},\ -\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

2b. One of the roots w_1 satisfies the condition $\mathrm{Re}\left(w_1\right) = 0$. *[3 marks]*

Given that $w_1 = \frac{z}{z - \mathrm{i}}$, express z in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

Markscheme

$w_1 = -2\mathrm{i}$

$\frac{z}{z-\mathrm{i}} = -2\mathrm{i}$ ***M1***

$z = -2\mathrm{i}\left(z - \mathrm{i}\right)$

$z\left(1 + 2\mathrm{i}\right) = -2$

$z = \frac{-2}{1+2\mathrm{i}}$ ***A1***

$z = -\frac{2}{5} + \frac{4}{5}\mathrm{i}$ ***A1***

Note: Accept $a = -\frac{2}{5}$, $b = \frac{4}{5}$.

[3 marks]

3a. Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg\left(z\right) < \frac{\pi}{2}$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where r, $\theta \in \mathbb{R}^{+}$. *[5 marks]*

Markscheme

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$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^{24} = 1\left(\cos 0 + \mathrm{i}\sin 0\right)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \frac{\pi n}{12}, \left(n \in \mathbb{Z}\right)$ ***(A1)***

$0 < \arg\left(z\right) < \frac{\pi}{2} \Rightarrow n =$ 1, 2, 3, 4, 5

$z = \mathrm{e}^{\frac{\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi\mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

Let *S* be the sum of the roots found in part (a).

3b. Show that Re *S* = Im *S*. *[4 marks]*

Markscheme

Re $S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

Im $S = \sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1***

Note: Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$, $\sin\frac{4\pi}{12} = \cos\frac{2\pi}{12}$, $\sin\frac{3\pi}{12} = \cos\frac{3\pi}{12}$, $\sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

⇒ Re S = Im S ***AG***

Note: Accept a geometrical method.

[4 marks]

3c. By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of cos $\frac{\pi}{12}$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are integers to be determined. *[3 marks]*

Markscheme

$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1***

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

[3 marks]

3d. Hence, or otherwise, show that $S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$. *[4 marks]*

Markscheme

$\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

Re S = $\cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

Re S = $\frac{\sqrt{2}+\sqrt{6}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1***

$= \frac{1}{2}\left(\sqrt{6} + 1 + \sqrt{2} + \sqrt{3}\right)$ ***A1***

$= \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)$

S = Re(S)(1 + i) since Re S = Im S, ***R1***

S = $\frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$ ***AG***

[4 marks]

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

4a. Express w^2 and w^3 in modulus-argument form. *[3 marks]*

Markscheme

$w^2 = 4\mathrm{cis}\left(\frac{2\pi}{3}\right)$; $w^3 = 8\mathrm{cis}\left(\pi\right)$ ***(M1)A1A1***

Note: Accept Euler form.

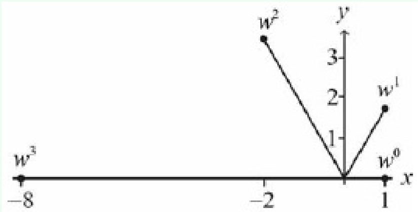
Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

[3 marks]

4b. Sketch on an Argand diagram the points represented by w^0 , w^1 , w^2 and w^3. *[2 marks]*

Markscheme



A1A1

[2 marks]

These four points form the vertices of a quadrilateral, *Q*.

4c. Show that the area of the quadrilateral *Q* is $\frac{21\sqrt{3}}{2}$. *[3 marks]*

Markscheme

use of area = $\frac{1}{2}ab\sin C$ ***M1***

$\frac{1}{2}\times 1\times 2\times \sin\frac{\pi}{3}+\frac{1}{2}\times 2\times 4\times \sin\frac{\pi}{3}+\frac{1}{2}\times 4\times 8\times \sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C=\frac{\pi}{3}$, ***A1*** for correct moduli.

$=\frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

4d. Let $z=2\left(\cos\frac{\pi}{n}+\mathrm{i}\sin\frac{\pi}{n}\right),\ n\in\mathbb{Z}^+$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2,\ \ldots,\ z^n$ form the vertices of a polygon P_n. *[6 marks]*

Show that the area of the polygon P_n can be expressed in the form $a\left(b^n-1\right)\sin\frac{\pi}{n}$, where $a,\ b\in\mathbb{R}$.

Markscheme

$\frac{1}{2} \times 2^0 \times 2^1 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^1 \times 2^2 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^2 \times 2^3 \times \sin\frac{\pi}{n} + \ldots + \frac{1}{2} \times 2^{n-}$ ***M1A1***

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n} \times \left(2^0 + 2^2 + 2^4 + \ldots + 2^{n-2}\right)$

identifying a geometric series with common ratio $2^2 (= 4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4} \times \sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}(4^n - 1)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

Consider the complex number $z = \frac{2+7\mathrm{i}}{6+2\mathrm{i}}$.

5a. Express z in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{Q}$. *[2 marks]*

Markscheme

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$z = \frac{(2+7\mathrm{i})}{(6+2\mathrm{i})} \times \frac{(6-2\mathrm{i})}{(6-2\mathrm{i})}$ ***(M1)***

$= \frac{26+38\mathrm{i}}{40} = \left(\frac{13+19\mathrm{i}}{20} = 0.65 + 0.95\mathrm{i}\right)$ ***A1***

[2 marks]

5b. Find the exact value of the modulus of z. *[2 marks]*

Markscheme

attempt to use $|z| = \sqrt{a^2 + b^2}$ ***(M1)***

$|z| = \sqrt{\frac{53}{40}} \left(= \frac{\sqrt{530}}{20}\right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

5c. Find the argument of z, giving your answer to 4 decimal places. *[2 marks]*

Markscheme

EITHER

arg z = arg(2 + 7i) − arg(6 + 2i) ***(M1)***

OR

arg z = arctan$\left(\frac{19}{13}\right)$ ***(M1)***

THEN

arg z = 0.9707 (radians) (= 55.6197 degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

Consider the complex numbers $z_1 = 1 + \sqrt{3}\mathrm{i}$,$z_2 = 1 + \mathrm{i}$ and $w = \frac{z_1}{z_2}$.

6a. By expressing z_1 and z_2 in modulus-argument form write down the modulus of w; *[3 marks]*

Markscheme

$z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$|w| = \sqrt{2}$ ***A1***

[3 marks]

6b. By expressing z_1 and z_2 in modulus-argument form write down the argument of w. *[1 mark]*

Markscheme

$z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$\arg w = \frac{\pi}{12}$ ***A1***

Notes: Allow ***FT*** from incorrect answers for z_1 and z_2 in modulus-argument form.

[1 mark]

6c. Find the smallest positive integer value of n, such that w^n is a real number. *[2 marks]*

Markscheme

EITHER

$\sin\left(\frac{\pi n}{12}\right) = 0$ ***(M1)***

OR

$\arg(w^n) = \pi$ ***(M1)***

$\frac{n\pi}{12} = \pi$

THEN

$\therefore n = 12$ ***A1***

[2 marks]