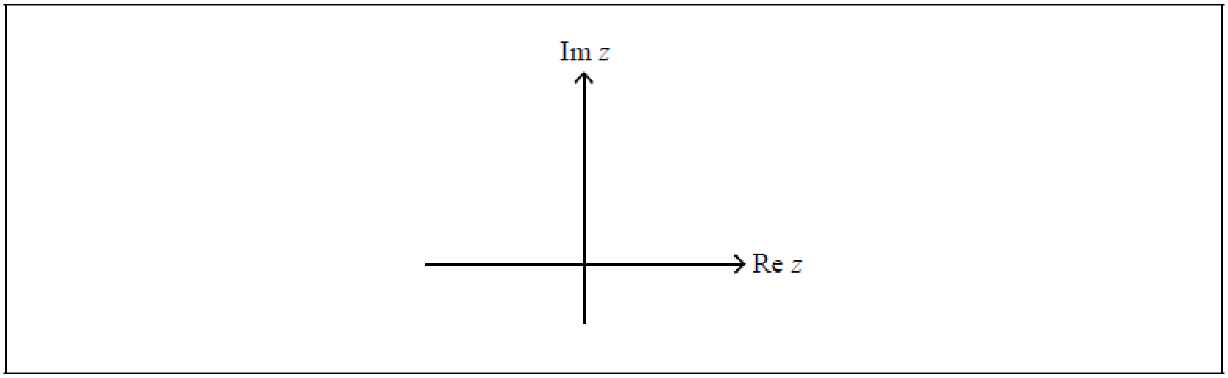
Complex numbers 21.04 *[56 marks]*

Let $z = a + b\mathrm{i}$, a, $\mathrm{b} \in \mathbb{R}^+$ and let $\arg z = \theta$.

1a. Show the points represented by z and $z - 2a$ on the following Argand diagram. *[1 mark]*



1b. Find an expression in terms of θ for $\arg(z - 2a)$. *[1 mark]*

1c. Find an expression in terms of θ for $\arg\left(\frac{z}{z-2a}\right)$. *[2 marks]*

1d. Hence or otherwise find the value of θ for which $\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0$. *[3 marks]*

2a. Find the roots of the equation $w^3 = 8\mathrm{i}$, $w \in \mathbb{C}$. Give your answers in Cartesian form. *[4 marks]*

2b. One of the roots w_1 satisfies the condition $\mathrm{Re}(w_1) = 0$. *[3 marks]*

Given that $w_1 = \frac{z}{z-\mathrm{i}}$, express z in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

3a. Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where r, $\theta \in \mathbb{R}^+$. *[5 marks]*

Let S be the sum of the roots found in part (a).

3b. Show that $\text{Re}\,S = \text{Im}\,S$. *[4 marks]*

3c. By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of $\cos\frac{\pi}{12}$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are integers to be determined. *[3 marks]*

3d. Hence, or otherwise, show that $S = \frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$. *[4 marks]*

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

4a. Express w^2 and w^3 in modulus-argument form. *[3 marks]*

4b. Sketch on an Argand diagram the points represented by w^0 , w^1 , w^2 and w^3. *[2 marks]*

These four points form the vertices of a quadrilateral, Q.

4c. Show that the area of the quadrilateral Q is $\frac{21\sqrt{3}}{2}$. *[3 marks]*

4d. Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right),\ n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2, \ldots,\ z^n$ form the vertices of a polygon P_n. *[6 marks]*

Show that the area of the polygon P_n can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a,\ b \in \mathbb{R}$.

Consider the complex number $z = \frac{2+7\mathrm{i}}{6+2\mathrm{i}}$.

5a. Express z in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{Q}$. *[2 marks]*

5b. Find the exact value of the modulus of z. *[2 marks]*

5c. Find the argument of z, giving your answer to 4 decimal places. *[2 marks]*

Consider the complex numbers $z_1 = 1 + \sqrt{3}\mathrm{i}$, $z_2 = 1 + \mathrm{i}$ and $w = \frac{z_1}{z_2}$.

6a. By expressing z_1 and z_2 in modulus-argument form write down the modulus of w; *[3 marks]*

6b. By expressing z_1 and z_2 in modulus-argument form write down the argument of w. *[1 mark]*

6c. Find the smallest positive integer value of n, such that w^n is a real number. *[2 marks]*