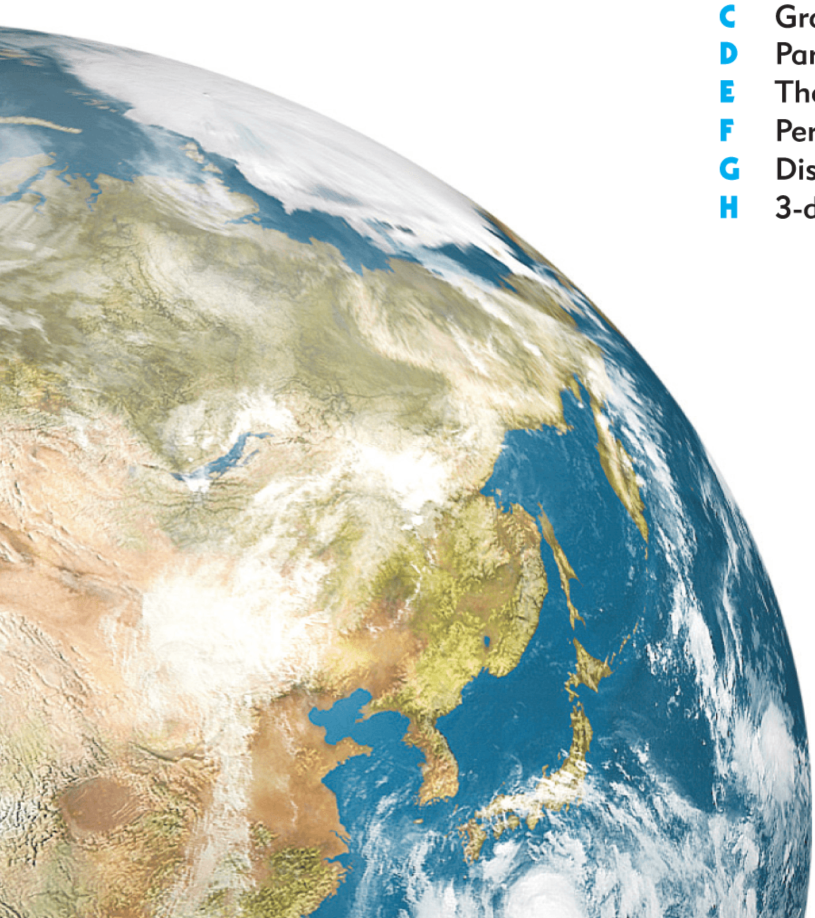
Chapter 6

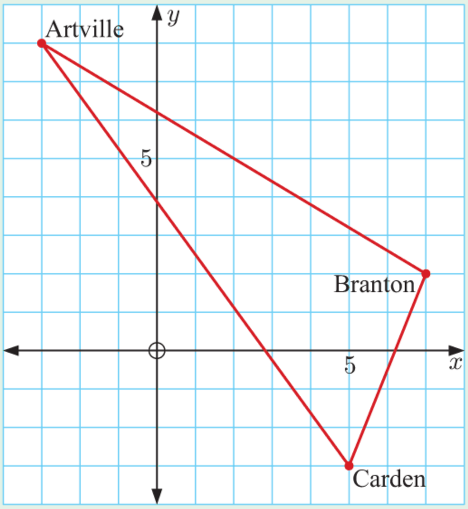
Coordinate geometry

Contents:



OPENING PROBLEM

The towns Artville, Branton, and Carden are joined by straight roads. On a road map Artville is at $(-3, 8)$, Branton is at $(7, 2)$, and Carden is at $(5, -3)$. The grid units are kilometres.



Things to think about:

- **a** How far is it from Artville to Branton?
- **b** What point is halfway between Branton and Carden?
- **c** Are any of the roads perpendicular to each other?
- **d**
  - **i** Can you find the *equation* of the road connecting Artville and Carden?
  - **ii** Does the point $(2, 1)$ lie on this road?

HISTORICAL NOTE

History shows that the two Frenchmen **René Descartes** and **Pierre de Fermat** arrived at the idea of **analytical geometry** at about the same time. Descartes' work *La Geometrie* was published first, in 1637, while Fermat's *Introduction to Loci* was not published until after his death.

Today, they are considered the co-founders of this important branch of mathematics which links algebra and geometry.

René Descartes

Pierre de Fermat

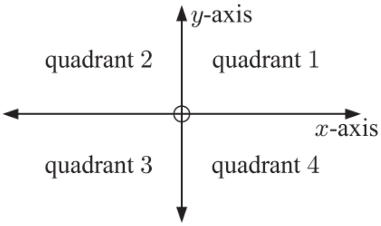
The initial approaches used by these mathematicians were quite opposite. Descartes began with a line or curve and then found the equation which described it. Fermat, to a large extent, started with an equation and investigated the shape of the curve it described.

Analytical geometry and its use of coordinates enabled **Isaac Newton** to later develop another important branch of mathematics called **calculus**. Newton humbly stated: "*If I have seen further than Descartes, it is because I have stood on the shoulders of giants.*"

The **number plane** consists of two perpendicular axes which intersect at the **origin**, O.

The ***x*-axis** is horizontal and the ***y*-axis** is vertical.

The axes divide the number plane into four **quadrants**.



The number plane is also known as either the **2-dimensional plane**, or the **Cartesian plane** after **René Descartes**.

The position of any point in the number plane can be specified in terms of an **ordered pair** of numbers (x, y), where:

- x is the **horizontal step** from O, and is the x-coordinate of the point
- y is the **vertical step** from O, and is the y-coordinate of the point.

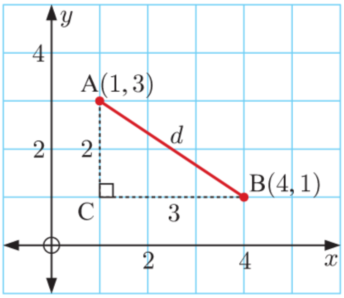
DEMO

A THE DISTANCE BETWEEN TWO POINTS

Suppose we want to find the distance d between the points A$(1, 3)$ and B$(4, 1)$.

By drawing line segments [AC] and [BC] along the grid lines, we form a right angled triangle with hypotenuse [AB].

$$\therefore \quad d^2 = 2^2 + 3^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad d^2 = 13$$
$$\therefore \quad d = \sqrt{13} \quad \{\text{as } d > 0\}$$



So, the distance between A and B is $\sqrt{13}$ units.

While this approach is effective, it is time-consuming because a diagram is needed.

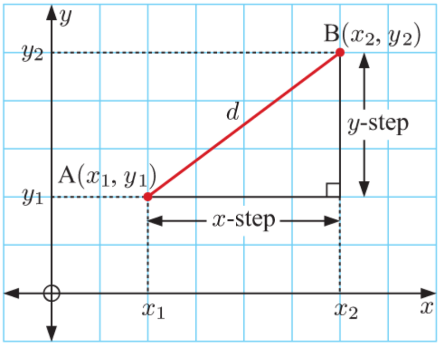
To make the process quicker, we can develop a formula.

To go from A(x_1, y_1) to B(x_2, y_2), we find the

$$x\text{-step} = x_2 - x_1$$
and $$y\text{-step} = y_2 - y_1.$$

Using Pythagoras' theorem,

$$(\text{AB})^2 = (x\text{-step})^2 + (y\text{-step})^2$$
$$\therefore \quad \text{AB} = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}$$
$$\therefore \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The distance d between two points (x_1, y_1) and (x_2, y_2) is given by

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Example 1 **Self Tutor**

Find the distance between A$(-2, 1)$ and B$(3, 4)$.

A$(-2, 1)$ B$(3, 4)$

x_1 y_1 x_2 y_2

$$\text{AB} = \sqrt{(3 - -2)^2 + (4 - 1)^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34} \text{ units}$$

The distance formula saves us having to graph the points each time we want to find a distance.

Example 2 **Self Tutor**

Consider the triangle formed by the points A$(1, 2)$, B$(2, 5)$, and C$(4, 1)$.

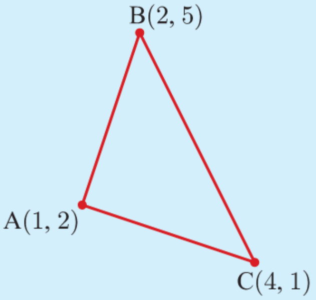
a Use the distance formula to classify the triangle as equilateral, isosceles, or scalene.

b Determine whether the triangle is right angled.

a

$$\begin{aligned} AB &= \sqrt{(2-1)^2+(5-2)^2} \\ &= \sqrt{1^2+3^2} \\ &= \sqrt{10} \text{ units} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(4-1)^2+(1-2)^2} \\ &= \sqrt{3^2+(-1)^2} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2+(1-5)^2} \\ &= \sqrt{2^2+(-4)^2} \\ &= \sqrt{20} \text{ units} \end{aligned}$$



Since AB = AC, the triangle is isosceles.

b The shortest sides are [AB] and [AC].

$$\begin{aligned} \text{Now} \quad AB^2 + AC^2 &= 10 + 10 \\ &= 20 \\ &= BC^2 \end{aligned}$$

Using the converse of Pythagoras' theorem, the triangle is right angled. The right angle is at A, opposite the longest side.

EXERCISE 6A

1 Find the distance between:

a A$(3, 1)$ and B$(5, 3)$ **b** C$(-1, 2)$ and D$(6, 2)$ **c** O$(0, 0)$ and P$(-2, 4)$

d E$(8, 0)$ and F$(2, -3)$ **e** G$(0, -2)$ and H$(0, 5)$ **f** I$(2, 0)$ and J$(0, -1)$

g R$(1, 2)$ and S$(-2, 3)$ **h** W$(1, -1)$ and Z$(\frac{1}{2}, -2)$.

2 In the map below, the grid lines are 10 km apart.

© OpenStreetMap contributors

Find the direct distance between:

a Dalgety Bay and Edinburgh **b** Coatbridge and Dalgety Bay **c** Coatbridge and Edinburgh.

3 Use the distance formula to classify triangle ABC as either equilateral, isosceles, or scalene:

- **a** A$(3, -1)$, B$(1, 8)$, C$(-6, 1)$
- **b** A$(1, 0)$, B$(3, 1)$, C$(4, 5)$
- **c** A$(-1, 0)$, B$(2, -2)$, C$(4, 1)$
- **d** A$(\sqrt{2}, 0)$, B$(-\sqrt{2}, 0)$, C$(0, -\sqrt{5})$
- **e** A$(\sqrt{3}, 1)$, B$(-\sqrt{3}, 1)$, C$(0, -2)$
- **f** A(a, b), B$(-a, b)$, C$(0, 2)$

4 Determine whether the following triangles are right angled. If there is a right angle, state the vertex where it occurs.

- **a** A$(-2, -1)$, B$(3, -1)$, C$(3, 3)$
- **b** A$(-1, 2)$, B$(4, 1)$, C$(4, -5)$
- **c** A$(1, -2)$, B$(3, 0)$, C$(-3, 2)$
- **d** A$(3, -4)$, B$(-2, -5)$, C$(-1, 1)$

Example 3

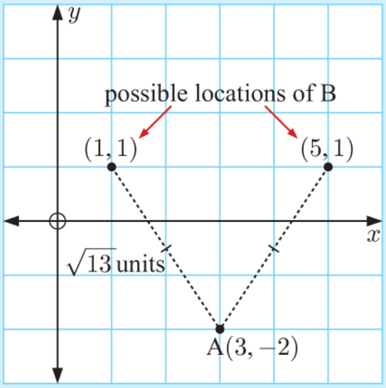
Self Tutor

Find b given that A$(3, -2)$ and B$(b, 1)$ are $\sqrt{13}$ units apart.
Explain your result using a diagram.

From A to B, x-step $= b - 3$
y-step $= 1 - -2 = 3$

$$\therefore \quad \sqrt{(b-3)^2 + 3^2} = \sqrt{13}$$
$$\therefore \quad (b-3)^2 + 9 = 13 \quad \{\text{squaring both sides}\}$$
$$\therefore \quad (b-3)^2 = 4$$
$$\therefore \quad b - 3 = \pm 2$$
$$\therefore \quad b = 3 \pm 2$$
$$\therefore \quad b = 5 \text{ or } 1$$

Point B could be at two possible locations:
$(5, 1)$ or $(1, 1)$.



5 For each of the cases below, find a and explain the result using a diagram:

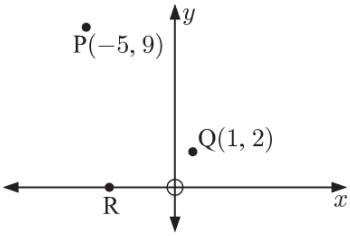
- **a** P$(2, 3)$ and Q$(a, -1)$ are 4 units apart
- **b** P$(-1, 1)$ and Q$(a, -2)$ are 5 units apart
- **c** X(a, a) is $\sqrt{8}$ units from the origin
- **d** A$(0, a)$ is equidistant from P$(3, -3)$ and Q$(-2, 2)$.

6 **a** Find the relationship between x and y if the point P(x, y) is always:

- **i** 3 units from O$(0, 0)$
- **ii** 2 units from A$(1, 3)$.

b Illustrate and describe the set $\{(x, y) \mid x^2 + y^2 = 1\}$.

7 P is at $(-5, 9)$, Q is at $(1, 2)$, and R is on the x-axis.
Given that triangle PQR is isosceles, find the possible coordinates of R.



B MIDPOINTS

The **midpoint** of line segment [AB] is the point which lies midway between points A and B.

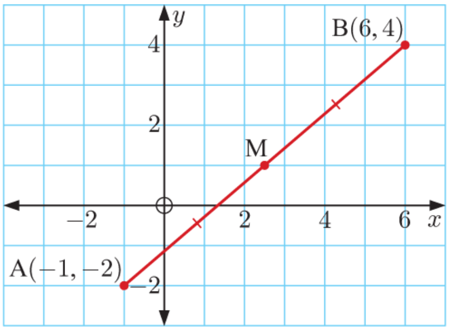
Consider the points A$(-1, -2)$ and B$(6, 4)$. From the diagram we see that the midpoint of [AB] is M$(2\frac{1}{2}, 1)$.

The x-coordinate of M is the *average* of the x-coordinates of A and B.

$\therefore$ the x-coordinate of M $= \dfrac{-1+6}{2} = \frac{5}{2} = 2\frac{1}{2}$

The y-coordinate of M is the *average* of the y-coordinates of A and B.

$\therefore$ the y-coordinate of M $= \dfrac{-2+4}{2} = 1$



If A(x_1, y_1) and B(x_2, y_2) are two points, then the **midpoint** of [AB] has coordinates $\left(\dfrac{\boldsymbol{x_1+x_2}}{\mathbf{2}}, \dfrac{\boldsymbol{y_1+y_2}}{\mathbf{2}}\right)$.

DEMO

Example 4 **Self Tutor**

Find the midpoint of [AB] given A$(-1, 3)$ and B$(4, 7)$.

$$\text{A}(\underset{\underset{x_1}{\uparrow}}{-1},\ \underset{\underset{y_1}{\uparrow}}{3}) \qquad \text{B}(\underset{\underset{x_2}{\uparrow}}{4},\ \underset{\underset{y_2}{\uparrow}}{7})$$

The x-coordinate of the midpoint $= \dfrac{x_1+x_2}{2} = \dfrac{-1+4}{2} = \frac{3}{2} = 1\frac{1}{2}$

The y-coordinate of the midpoint $= \dfrac{y_1+y_2}{2} = \dfrac{3+7}{2} = 5$

So, the midpoint is $(1\frac{1}{2}, 5)$.

EXERCISE 6B

1 Find the coordinates of the midpoint of the line segment joining:

- **a** $(8, 1)$ and $(2, 5)$
- **b** $(2, -3)$ and $(0, 1)$
- **c** $(3, 0)$ and $(0, 6)$
- **d** $(-1, 4)$ and $(1, 4)$
- **e** $(5, -3)$ and $(-1, 0)$
- **f** $(5, 9)$ and $(-3, -4)$.

Example 5 **Self Tutor**

M is the midpoint of [AB]. A is $(1, 3)$ and M is $(4, -2)$. Find the coordinates of B.

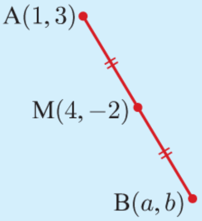
Suppose B has coordinates (a, b).

$\therefore \dfrac{a+1}{2} = 4$ and $\dfrac{b+3}{2} = -2$

$\therefore a+1 = 8$ and $b+3 = -4$

$\therefore a = 7$ and $b = -7$

$\therefore$ B is $(7, -7)$.



Example 6

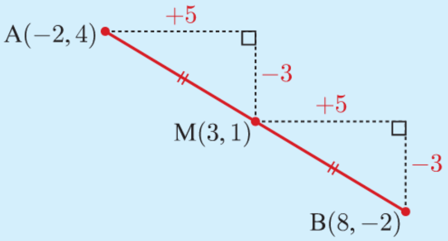
Self Tutor

Suppose A is $(-2,\ 4)$ and M is $(3,\ 1)$, where M is the midpoint of [AB].
Use *equal steps* to find the coordinates of B.

x-step: $-2 \xrightarrow{+5} 3 \xrightarrow{+5} 8$

y-step: $4 \xrightarrow{-3} 1 \xrightarrow{-3} -2$

$\therefore$ B is $(8,\ -2)$.



2 M is the midpoint of [AB]. Find the coordinates of B for:

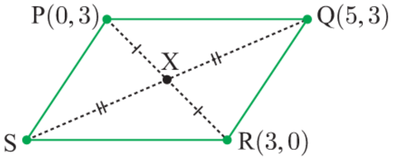
- **a** A(6, 4) and M(3, -1)
- **b** A(-5, 0) and M(0, -1)
- **c** A(3, -2) and M($1\frac{1}{2}$, 2)
- **d** A(-1, -2) and M($-\frac{1}{2}$, $2\frac{1}{2}$)
- **e** A(7, -3) and M(0, 0)
- **f** A(3, -1) and M(0, $-\frac{1}{2}$).

Check your answers using the *equal steps* method given in **Example 6**.

3 [AB] is a diameter of a circle with centre C. If A is (3, -2) and B is (-1, -4), find the coordinates of C.

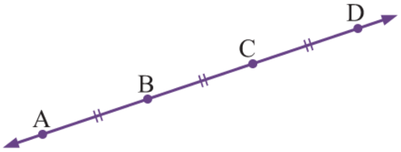
4 [PQ] is a diameter of a circle with centre (3, $-\frac{1}{2}$). If Q is (-1, 2), find the coordinates of P.

5 The diagonals of parallelogram PQRS bisect each other at X. Find the coordinates of S.



6 Triangle ABC has vertices A(-1, 3), B(1, -1), and C(5, 2). Find the length of the line segment from A to the midpoint of [BC].

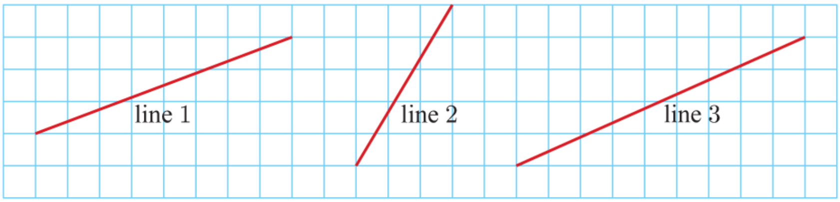
7 A, B, C, and D are four points on the same straight line. The distances between successive points are equal, as shown. If A is (1, -3), C is (4, a), and D is (b, 5), find the values of a and b.



8 The midpoints of the sides of a triangle are (5, 4), (8, 5), and (6, 0). Find the coordinates of the vertices of the triangle.

C GRADIENT

Consider the lines shown:



We can see that line 2 rises much faster than the other two lines, so line 2 is steepest.

However, most people would find it hard to tell which of lines 1 and 3 is steeper just by looking at them. We therefore need a more precise way to measure the steepness of a line.

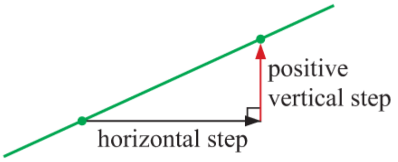
The **gradient** of a line is a measure of its steepness.

To calculate the gradient of a line, we first choose any two distinct points on the line. We can move from one point to the other by making a positive **horizontal step** followed by a **vertical step**.

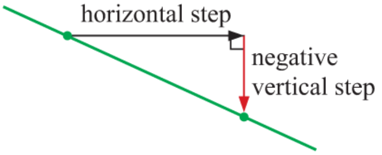
The gradient is calculated by dividing the vertical step by the horizontal step.

The **gradient** of a line $= \dfrac{\text{vertical step}}{\text{horizontal step}}$ or $\dfrac{y\text{-step}}{x\text{-step}}$.

If the line is sloping upwards, then both steps are positive, so the line has a **positive gradient**.



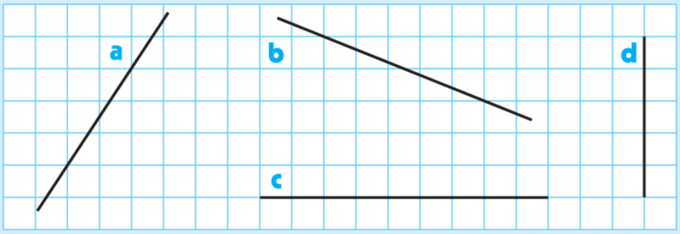
If the line is sloping downwards, the horizontal step is positive and the vertical step is negative, so the line has a **negative gradient**.

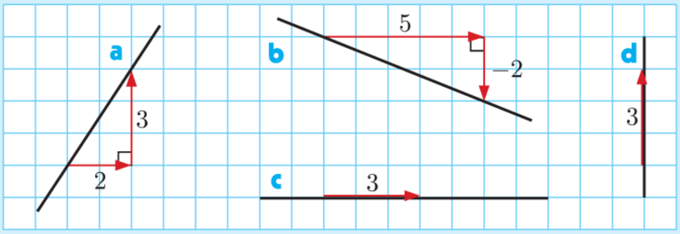


Example 7

Self Tutor

Find the gradient of each line segment:





a gradient $= \frac{3}{2}$

b gradient $= \frac{-2}{5} = -\frac{2}{5}$

c gradient $= \frac{0}{3} = 0$

d gradient $= \frac{3}{0}$ which is undefined

From the previous **Example**, we can see that:

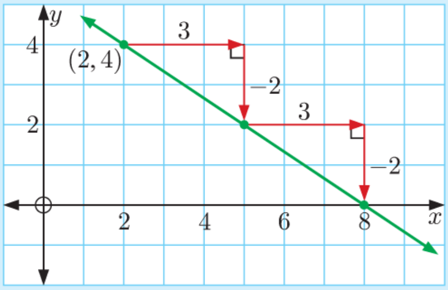
- The gradient of all **horizontal** lines is **0**, since the vertical step is 0.
- The gradient of all **vertical** lines is **undefined**, since the horizontal step is 0.

Example 8 **Self Tutor**

Draw a line with gradient $-\frac{2}{3}$, through the point (2, 4).

Plot the point (2, 4).

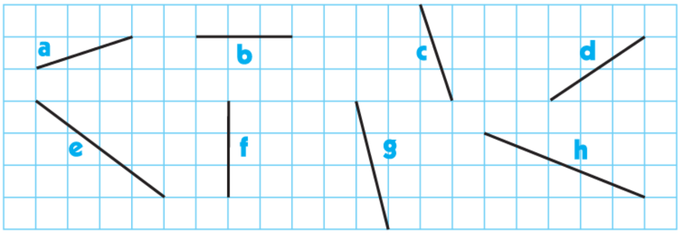
$$\text{gradient} = -\frac{2}{3} = \frac{-2}{3} \begin{matrix} \leftarrow y\text{-step} \\ \leftarrow x\text{-step} \end{matrix}$$




Use a positive x-step.

EXERCISE 6C.1

1 Find the gradient of each line segment:



2 On grid paper, draw a line segment with gradient:

a $\frac{3}{4}$ **b** $-\frac{1}{2}$ **c** 2 **d** -3 **e** 0 **f** $-\frac{2}{5}$

3 Draw a line with gradient $\frac{1}{2}$, through the point $(3, -1)$.

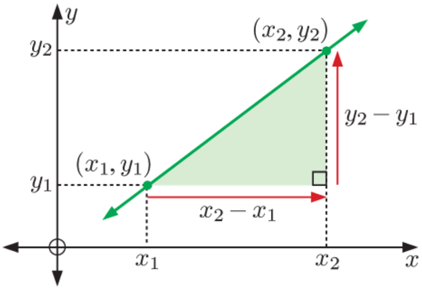
4 Draw a line with gradient $-\frac{3}{4}$, through the point $(-1, 3)$.

5 On the same set of axes, draw lines through (2, 3) with gradients $\frac{1}{3}$, $\frac{3}{4}$, 2, and 4.

6 On the same set of axes, draw lines through $(-1, 2)$ with gradients 0, $-\frac{2}{5}$, -2, and -5.

THE GRADIENT FORMULA

The **gradient** of the line through (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$.



Example 9

Self Tutor

Find the gradient of the line through $(3,\ -2)$ and $(6,\ 4)$.

$(\underset{x_1}{\underset{\uparrow}{3}},\ \underset{y_1}{\underset{\uparrow}{-2}})$ $\quad$ $(\underset{x_2}{\underset{\uparrow}{6}},\ \underset{y_2}{\underset{\uparrow}{4}})$

$$\begin{aligned} \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - -2}{6 - 3} \\ &= \tfrac{6}{3} \\ &= 2 \end{aligned}$$

EXERCISE 6C.2

1 Use the gradient formula to find the gradient of the line through $A(-2,\ -3)$ and $B(5,\ 1)$. Plot the line segment [AB] on a set of axes to illustrate your answer.

2 Find the gradient of the line segment joining:

a $(2,\ 3)$ and $(7,\ 4)$
b $(5,\ 7)$ and $(1,\ 6)$
c $(1,\ -2)$ and $(3,\ 6)$
d $(5,\ 5)$ and $(-1,\ 5)$
e $(3,\ -1)$ and $(3,\ -4)$
f $(5,\ -1)$ and $(-2,\ -3)$
g $(-5,\ 2)$ and $(2,\ 0)$
h $(0,\ -1)$ and $(-2,\ -3)$
i $(-1,\ 7)$ and $(11,\ -9)$.

Example 10

Self Tutor

Find t given that the line segment joining $(5,\ -2)$ and $(9,\ t)$ has gradient $\frac{2}{3}$.

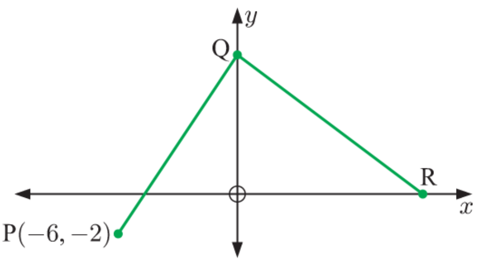
The line segment joining $(5,\ -2)$ and $(9,\ t)$ has gradient $= \dfrac{t - -2}{9 - 5} = \dfrac{t + 2}{4}$.

$$\begin{aligned} \therefore\ \frac{t+2}{4} &= \frac{2}{3} \\ \therefore\ 3(t+2) &= 8 \\ \therefore\ 3t + 6 &= 8 \\ \therefore\ 3t &= 2 \\ \therefore\ t &= \tfrac{2}{3} \end{aligned}$$

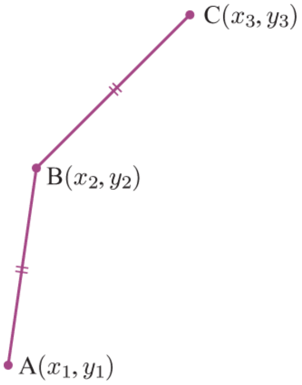
3 Find t given that the line segment joining:

a $(-3,\ 5)$ and $(4,\ t)$ has gradient 2
b $(5,\ t)$ and $(10,\ 12)$ has gradient $-\frac{1}{2}$
c $(3,\ -6)$ and $(t,\ -2)$ has gradient 3
d $(t,\ 9)$ and $(4,\ 7)$ has gradient $-\frac{3}{5}$
e $(2,\ 5)$ and $(t,\ t)$ has gradient $\frac{4}{7}$
f $(t,\ 2t)$ and $(-3,\ 12)$ has gradient $-\frac{1}{4}$.

4



The gradient of [PQ] is $\frac{3}{2}$, and the gradient of [QR] is $-\frac{3}{4}$. Find the coordinates of R.



5 $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are three points such that the gradient of [AB] is 7, the gradient of [BC] is 1, and $AB = BC$.

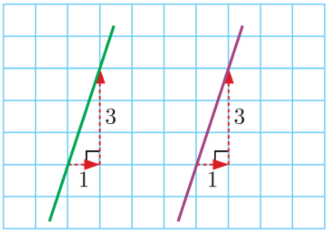
  a Use the gradient formula to show that $(y_2 - y_1)^2 = 49(x_2 - x_1)^2$ and $(y_3 - y_2)^2 = (x_3 - x_2)^2$.

  b Use **a** and the distance formula to show that $x_3 - x_2 = 5(x_2 - x_1)$.

  c Hence, find the gradient of [AC].

D PARALLEL AND PERPENDICULAR LINES

PARALLEL LINES



The given lines are parallel, and both of them have a gradient of 3.

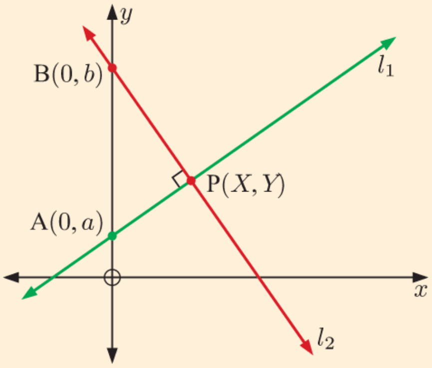
- If two lines are **parallel**, then they have **equal gradient**.
- If two lines have **equal gradient**, then they are **parallel**.

PERPENDICULAR LINES

INVESTIGATION PERPENDICULAR LINES

Consider two lines l_1 and l_2 which intersect at right angles at point $P(X, Y)$.

If l_1 and l_2 are not horizontal or vertical, then both lines will cut the y-axis. We suppose line l_1 cuts the y-axis at $A(0, a)$, and line l_2 cuts the y-axis at $B(0, b)$.



What to do:

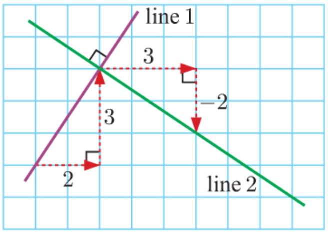
1 Explain why $(AP)^2 + (BP)^2 = (AB)^2$.

2 Hence show that $X^2 + (Y - a)^2 + X^2 + (Y - b)^2 = (b - a)^2$.

3 By expanding the brackets and simplifying, show that $Y^2 - (a + b)Y + ab = -X^2$.

4 Hence show that $\dfrac{Y - a}{X} \times \dfrac{Y - b}{X} = -1$.

5 Explain the significance of the result in **4**.



Line 1 and line 2 are perpendicular.

Line 1 has gradient $\frac{3}{2}$.

Line 2 has gradient $\frac{-2}{3} = -\frac{2}{3}$.

We see that the gradients are *negative reciprocals* of each other, and their product is $\frac{3}{2} \times -\frac{2}{3} = -1$.

For lines which are not horizontal or vertical:

- if the lines are **perpendicular**, then their gradients are **negative reciprocals**
- if the gradients are **negative reciprocals**, then the lines are **perpendicular**.

DEMO

Example 11 **Self Tutor**

Find the gradient of all lines perpendicular to a line with a gradient of:

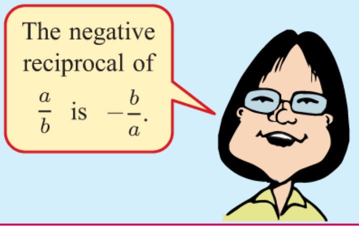
a $\frac{2}{7}$ **b** -5

a The negative reciprocal of $\frac{2}{7}$ is $-\frac{7}{2}$.

$\therefore$ the gradient of any perpendicular line is $-\frac{7}{2}$.

b The negative reciprocal of $-5 = \frac{-5}{1}$ is $\frac{1}{5}$.

$\therefore$ the gradient of any perpendicular line is $\frac{1}{5}$.



EXERCISE 6D.1

1 Find the gradient of all lines perpendicular to a line with a gradient of:

a $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** 3 **d** 7

e $-\frac{2}{5}$ **f** $-\frac{7}{2}$ **g** $-1\frac{1}{3}$ **h** -1

2 The gradients of two lines are listed below. Which of the line pairs are perpendicular?

a $\frac{1}{3}, 3$ **b** $5, -5$ **c** $\frac{3}{7}, -2\frac{1}{3}$ **d** $4, -\frac{1}{4}$

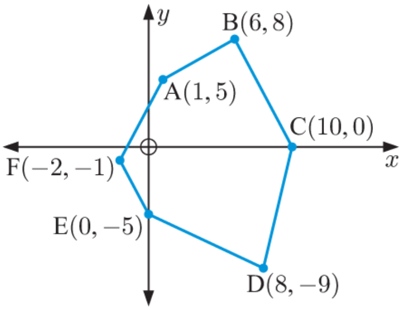
e $6, -\frac{5}{6}$ **f** $\frac{2}{3}, -\frac{3}{2}$ **g** $\frac{p}{q}, \frac{q}{p}$ **h** $\frac{a}{b}, -\frac{b}{a}$

3 Consider the hexagon alongside.

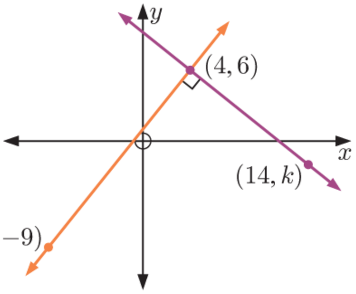
a Calculate the gradient of each side of the hexagon.

b Which sides are:

i parallel **ii** perpendicular?



4 Find the value of k:



5 Consider the points $A(1, 4)$, $B(-1, 0)$, $C(6, 3)$, and $D(t, -1)$. Find t if:

- **a** [AB] is parallel to [CD]
- **b** [AC] is parallel to [DB]
- **c** [AB] is perpendicular to [CD]
- **d** [AD] is perpendicular to [BC].

6 Consider the points $P(1, 5)$, $Q(5, 7)$, and $R(3, 1)$.

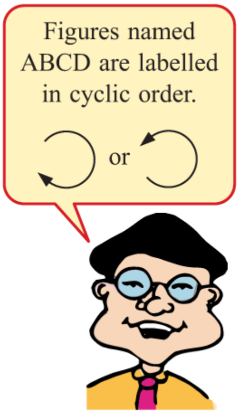
- **a** Show that triangle PQR is isosceles.
- **b** Find the midpoint M of [QR].
- **c** Use gradients to verify that [PM] is perpendicular to [QR].
- **d** Draw a sketch to illustrate what you have found.

7 For the points $A(-1, 1)$, $B(1, 5)$, and $C(5, 1)$, M is the midpoint of [AB], and N is the midpoint of [BC].

- **a** Show that [MN] is parallel to [AC].
- **b** Show that [MN] is half the length of [AC].

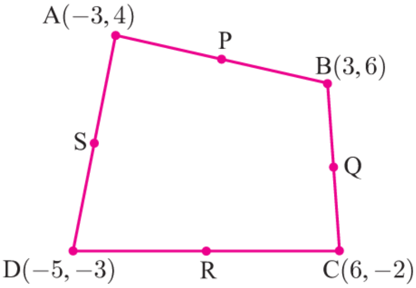
8 Consider the points $A(1, 3)$, $B(6, 3)$, $C(3, -1)$, and $D(-2, -1)$.

- **a** Use the distance formula to show that ABCD is a rhombus.
- **b** Find the midpoints of [AC] and [BD].
- **c** Show that [AC] and [BD] are perpendicular.
- **d** Draw a sketch to illustrate your findings.



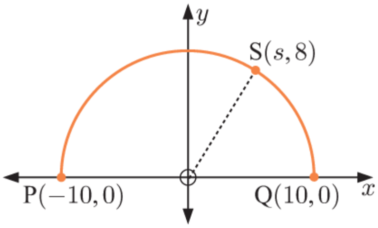
9 The sketch of quadrilateral ABCD is not drawn to scale. P, Q, R, and S are the midpoints of [AB], [BC], [CD], and [DA] respectively.

- **a** Find the coordinates of:
  - **i** P
  - **ii** Q
  - **iii** R
  - **iv** S.
- **b** Find the gradient of:
  - **i** [PQ]
  - **ii** [QR]
  - **iii** [RS]
  - **iv** [SP].
- **c** What can be deduced about quadrilateral PQRS?



10 $S(s, 8)$ lies on a semi-circle as shown.

- **a** Find s.
- **b** Find the gradient of:
  - **i** [PS]
  - **ii** [SQ].
- **c** Hence show that angle PSQ is a right angle.

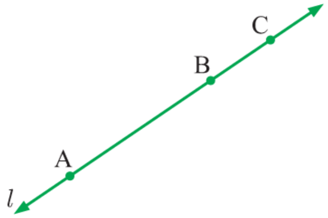


COLLINEAR POINTS

Three or more points are **collinear** if they lie on the same straight line.

Consider the three collinear points A, B, and C, which all lie on the line l.

gradient of [AB] = gradient of [BC] = gradient of l



Three points A, B, and C are **collinear** if gradient of [AB] = gradient of [BC].

Example 12 **Self Tutor**

Show that the points $A(1, -1)$, $B(6, 9)$, and $C(3, 3)$ are collinear.

Gradient of [AB] $= \frac{9 - -1}{6 - 1} = \frac{10}{5} = 2.$ Gradient of [BC] $= \frac{3 - 9}{3 - 6} = \frac{-6}{-3} = 2.$

$\therefore$ [AB] is parallel to [BC], and point B is common to both line segments.

$\therefore$ A, B, and C are collinear.

EXERCISE 6D.2

1 Determine whether the following sets of points are collinear:

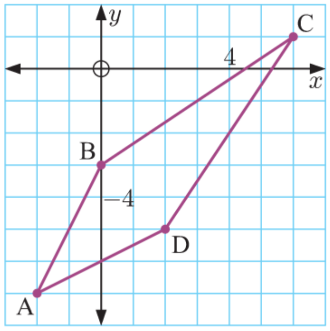
- **a** $A(1, 2)$, $B(4, 6)$, and $C(-4, -4)$
- **b** $P(-6, -6)$, $Q(-1, 0)$, and $R(4, 6)$
- **c** $R(5, 2)$, $S(-6, 5)$, and $T(0, -4)$
- **d** $A(0, -2)$, $B(-1, -5)$, and $C(3, 7)$.

2 Find c given that these three points are collinear:

- **a** $A(-4, -2)$, $B(0, 2)$, and $C(c, 5)$
- **b** $P(3, -2)$, $Q(4, c)$, and $R(-1, 10)$.

3 The points $A(-2, -7)$, $B(0, -3)$, $C(6, 1)$, and $D(2, -5)$ form a kite.

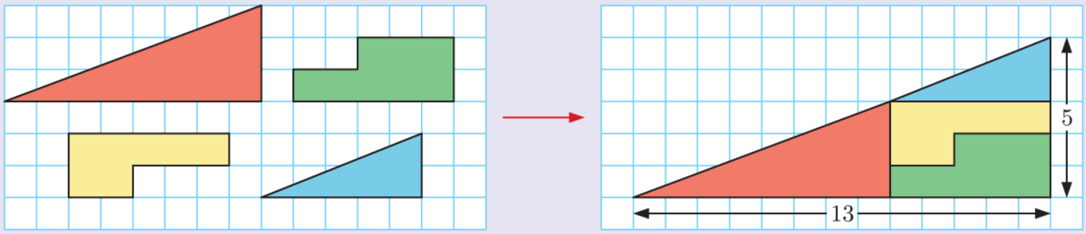
- **a** Find the midpoint M of [BD].
- **b** Show that A, M, and C are collinear.
- **c** Show that [AC] is perpendicular to [BD].



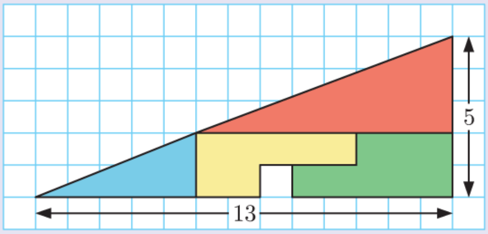
PUZZLE THE MISSING SQUARE

Stephanie presents the following puzzle to her friend Courtney:

"I can arrange these four shapes to form a right angled triangle which is 13 units long and 5 units high."



"I can then rearrange the shapes to form a triangle of the same size. However, there is now a missing square."



Can you explain why there is a missing square?

E THE EQUATION OF A LINE

The **equation of a line** is a rule which connects the x and y-coordinates of **all** points on the line.

The equation of a line is commonly written in either **gradient-intercept form** or in **general form**.

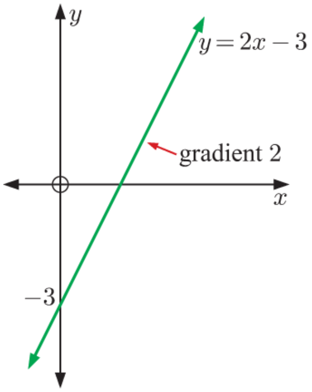
GRADIENT-INTERCEPT FORM

$y = mx + c$ is called the **gradient-intercept form** of an equation of a line.

The line with equation $y = mx + c$ has gradient m and y-intercept c.

For example, the line with equation $y = 2x - 3$ has gradient 2 and y-intercept -3.

The **y-intercept** of a line is the y-coordinate of the point where the line cuts the y-axis.



GENERAL FORM

$Ax + By = C$ is called the **general form** of the equation of a line.

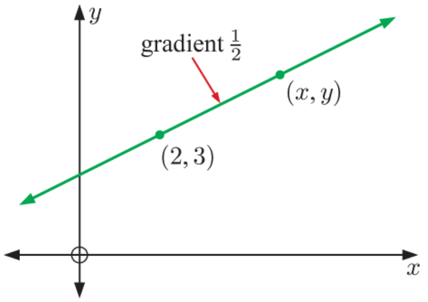
For example, the equations $2x + 3y = 5$ and $x - 6y = -7$ are in general form.

Equations in general form are usually written with a positive coefficient of x.

FINDING THE EQUATION OF A LINE

If we are given enough information about a line, we can determine its equation.

> To determine the equation of a line, we need to know either:
>
> - its gradient and at least one point which lies on the line, or
> - two points which lie on the line.



Suppose that a line has gradient $\frac{1}{2}$, and passes through the point $(2, 3)$.

For any point (x, y) which lies on the line, the gradient between $(2, 3)$ and (x, y) is $\dfrac{y-3}{x-2}$.

$\therefore$ the line has equation $\dfrac{y-3}{x-2} = \dfrac{1}{2}$

which can be written as $y - 3 = \frac{1}{2}(x-2)$.

We can rearrange this to find the equation of the line in either gradient-intercept form or general form:

Gradient-intercept form	**General form**
$y - 3 = \frac{1}{2}(x-2)$	$y - 3 = \frac{1}{2}(x-2)$
$\therefore\ y - 3 = \frac{1}{2}x - 1$	$\therefore\ 2(y-3) = 1(x-2)$
$\therefore\ y = \frac{1}{2}x + 2$	$\therefore\ 2y - 6 = x - 2$
	$\therefore\ x - 2y = -4$

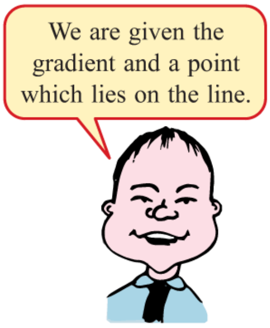
> If a straight line has gradient m and passes through (a, b), then it has equation
>
> $$\boldsymbol{\frac{y-b}{x-a} = m \quad \text{or} \quad y - b = m(x-a).}$$
>
> We can rearrange the equation into either **gradient-intercept form** or **general form**.

Example 13 **Self Tutor**

Find, in *gradient-intercept form*, the equation of the line with gradient 5 that passes through $(-1, 3)$.

The equation of the line is

$$\begin{aligned} y - 3 &= 5(x - -1) \\ \therefore\ y - 3 &= 5(x+1) \\ \therefore\ y - 3 &= 5x + 5 \\ \therefore\ y &= 5x + 8 \end{aligned}$$



EXERCISE 6E.1

1 Find, in *gradient-intercept form*, the equation of the line with:

- **a** gradient 2, passing through $(1, 3)$
- **b** gradient -1, passing through $(-1, 2)$
- **c** gradient $\frac{2}{3}$, passing through $(-3, 1)$
- **d** gradient $-\frac{4}{5}$, passing through $(4, -2)$
- **e** gradient $-\frac{3}{4}$, passing through $(6, -5)$.

Example 14 **Self Tutor**

Find, in *general form*, the equation of the line with gradient $\frac{3}{4}$ that passes through $(5, -2)$.

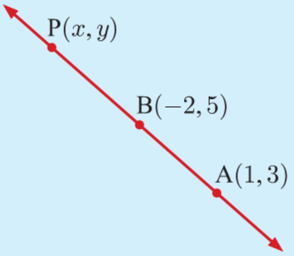
The equation of the line is $$y - -2 = \tfrac{3}{4}(x - 5)$$
$$\therefore\ 4(y + 2) = 3(x - 5)$$
$$\therefore\ 4y + 8 = 3x - 15$$
$$\therefore\ 3x - 4y = 23$$

2 Find, in *general form*, the equation of the line with:

- **a** gradient 4, passing through $(3, 5)$
- **b** gradient $-\frac{3}{5}$, passing through $(-2, 1)$
- **c** gradient $\frac{1}{3}$, passing through $(1, 4)$
- **d** gradient $-\frac{3}{4}$, passing through $(0, 6)$
- **e** gradient $\frac{2}{7}$, passing through $(-5, -5)$.

Example 15 **Self Tutor**

Find, in *gradient-intercept form*, the equation of the line which passes through A$(1, 3)$ and B$(-2, 5)$.



The line has gradient $= \dfrac{5-3}{-2-1} = \dfrac{2}{-3} = -\dfrac{2}{3}$,

and passes through the point A$(1, 3)$.

$\therefore$ the equation of the line is
$$y - 3 = -\tfrac{2}{3}(x - 1)$$
$$\therefore\ y - 3 = -\tfrac{2}{3}x + \tfrac{2}{3}$$
$$\therefore\ y = -\tfrac{2}{3}x + \tfrac{11}{3}$$

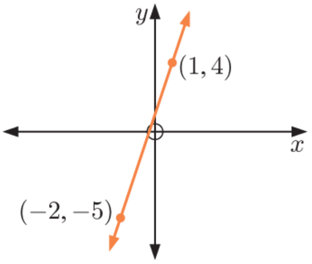
We could use *either* A or B as the point which lies on the line.

3 Find, in *gradient-intercept form*, the equation of the line which passes through:

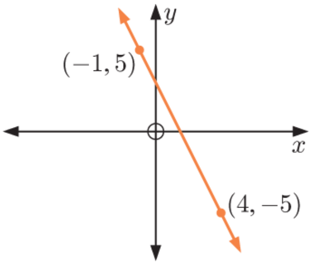
- **a** A$(8, 4)$ and B$(5, 1)$
- **b** A$(5, -1)$ and B$(4, 0)$
- **c** A$(-2, 4)$ and B$(-3, -2)$
- **d** P$(-4, 6)$ and Q$(2, 9)$
- **e** M$(-1, -2)$ and N$(5, -4)$
- **f** R$(2, -4)$ and S$(7, -7)$.

4 Find, in *general form*, the equation of each of the following lines:

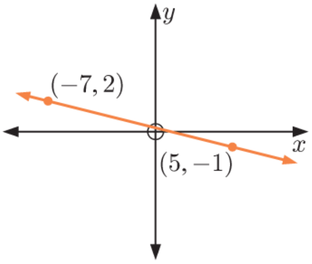
a



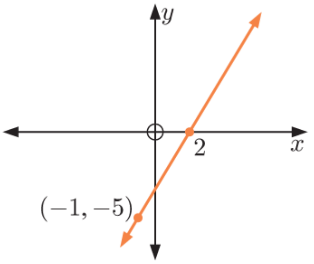
b



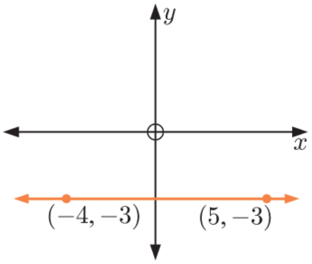
c



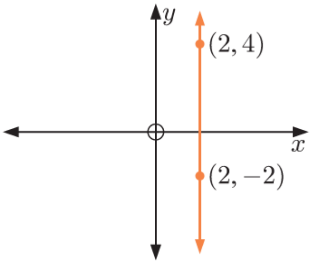
d



e



f



5 **a** Find, in general form, the equation of the line through A$(-3, 5)$ and B$(2, 1)$.

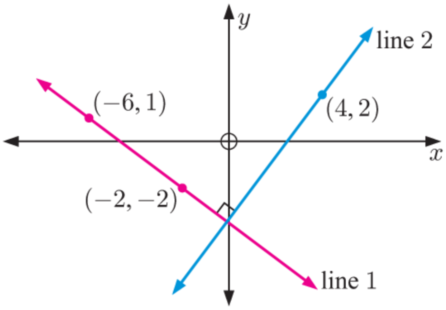
b Show that the point C$(12, -7)$ also lies on this line.

6 Find the equation of the line which:

- **a** cuts the x-axis at 5 and the y-axis at -2
- **b** cuts the x axis at -1, and passes through $(-3, 4)$
- **c** is parallel to a line with gradient 2, and passes through the point $(-1, 4)$
- **d** is perpendicular to a line with gradient $\frac{3}{4}$, and cuts the x-axis at 5
- **e** is perpendicular to a line with gradient -2, and passes through $(-2, 3)$.

7 **a** Find the gradient of line 1.

b Hence, find the equation of line 2.

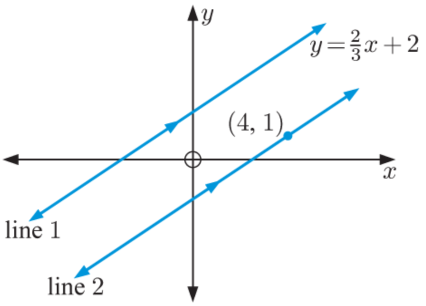


8 Find the equation of the line through $(-1, 7)$, which is parallel to the line through $(-3, -4)$ and $(2, 3)$.

9 Find the equation of the line through $(2, 0)$, which is perpendicular to the line through $(-5, 3)$ and $(4, -3)$.

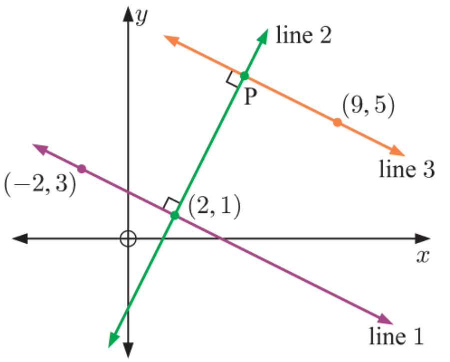
10 **a** Find, in gradient-intercept form, the equation of line 2.

b Hence, find the y-intercept of line 2.



11 Lines l_1 and l_2 are perpendicular to each other, and intersect at $(-2, 5)$. The equation of l_1 is $y = 3x + 11$. Find, in general form, the equation of l_2.

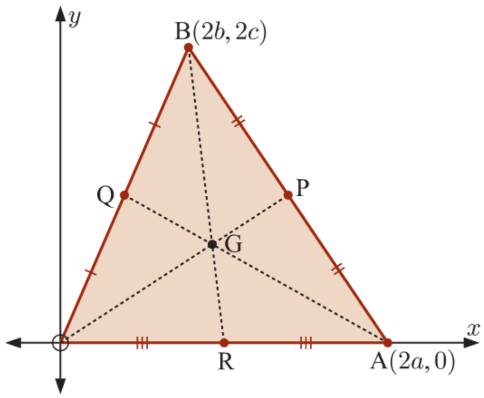
12



a Find, in gradient-intercept form, the equation of:
  i line 1 **ii** line 2 **iii** line 3.

b Show that the coordinates of P are $(5,\ 7)$.

13



A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

a Show that [OP] has equation $cx - (a+b)\,y = 0$.

b Show that [AQ] has equation $cx - (b-2a)y = 2ac$.

c Prove that the third median [BR] passes through the point of intersection G of medians [OP] and [AQ].

FINDING THE GENERAL FORM OF A LINE QUICKLY

If a line has gradient $\frac{3}{4}$, its equation has the form

$$y = \tfrac{3}{4}x + c$$
$$\therefore\ 4y = 3x + 4c$$
$$\therefore\ 3x - 4y = C \quad \text{for some constant } C.$$

Similarly, if a line has gradient $-\frac{3}{4}$, its equation has the form $3x + 4y = C$.

- The equation of a line with gradient $\dfrac{A}{B}$ has the general form $Ax - By = C$.
- The equation of a line with gradient $-\dfrac{A}{B}$ has the general form $Ax + By = C$.

The constant term C is obtained by substituting the coordinates of any point which lies on the line.

Example 16 **Self Tutor**

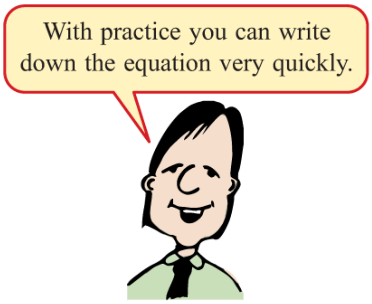
Find the equation of the line:

a with gradient $\frac{3}{4}$, which passes through $(5,\ -2)$

b with gradient $-\frac{3}{4}$, which passes through $(1,\ 7)$.

a The equation is $3x - 4y = 3(5) - 4(-2)$
$\therefore\ 3x - 4y = 23$

b The equation is $3x + 4y = 3(1) + 4(7)$
$\therefore\ 3x + 4y = 31$



EXERCISE 6E.2

1 Find the equation of the line:

- **a** through $(4, 1)$ with gradient $\frac{1}{2}$
- **b** through $(-2, 5)$ with gradient $\frac{2}{3}$
- **c** through $(5, 0)$ with gradient $\frac{3}{4}$
- **d** through $(3, -2)$ with gradient 3
- **e** through $(1, 4)$ with gradient $-\frac{1}{3}$
- **f** through $(2, -3)$ with gradient $-\frac{3}{4}$
- **g** through $(3, -2)$ with gradient -2
- **h** through $(0, 4)$ with gradient -3.

2 Find the gradient of the line with equation:

- **a** $2x + 3y = 8$
- **b** $3x - 7y = 11$
- **c** $6x - 11y = 4$
- **d** $5x + 6y = -1$
- **e** $3x + 6y = -1$
- **f** $15x - 5y = 17$

3 Explain why:

- **a** any line parallel to $3x + 5y = 2$ has the form $3x + 5y = C$
- **b** any line perpendicular to $3x + 5y = 2$ has the form $5x - 3y = C$.

4 Find the equation of the line which is:

- **a** parallel to the line $3x + 4y = 6$ and which passes through $(2, 1)$
- **b** perpendicular to the line $5x + 2y = 10$ and which passes through $(-1, -1)$
- **c** perpendicular to the line $x - 3y + 6 = 0$ and which passes through $(-4, 0)$
- **d** parallel to the line $x - 3y = 11$ and which passes through $(0, 0)$.

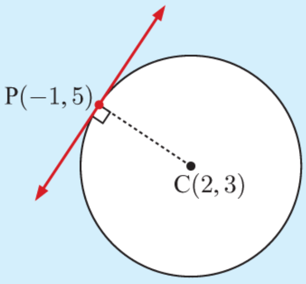
5 $2x - 3y = 6$ and $6x + ky = 4$ are two straight lines.

- **a** Write down the gradient of each line.
- **b** Find k such that the lines are parallel.
- **c** Find k such that the lines are perpendicular.

6 Answer the **Opening Problem** on page **104**.

Example 17 **Self Tutor**

A circle has centre $(2, 3)$. Find the equation of the tangent to the circle with point of contact $(-1, 5)$.



The gradient of [CP] is $\frac{5-3}{(-1)-2} = \frac{2}{-3}$

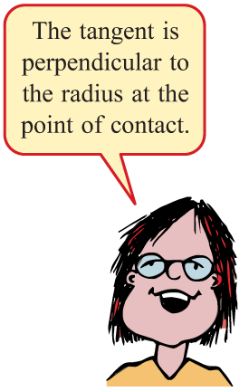
$= -\frac{2}{3}$

$\therefore$ the gradient of the tangent at P is $\frac{3}{2}$

$\therefore$ the equation of the tangent is

$3x - 2y = 3(-1) - 2(5)$

which is $3x - 2y = -13$.



7 Find the equation of the tangent to the circle:

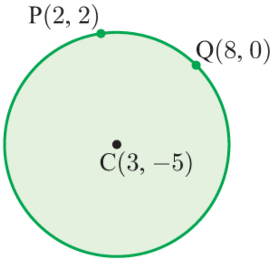
a with centre $(0,\ 2)$ if the point of contact is $(-1,\ 5)$

b with centre $(0,\ 0)$ if the point of contact is $(3,\ -2)$

c with centre $(3,\ -1)$ if the point of contact is $(-1,\ 1)$

d with centre $(2,\ -2)$ if the point of contact is $(5,\ -2)$.

8



a Find the equation of the tangent to the circle at:

i P **ii** Q.

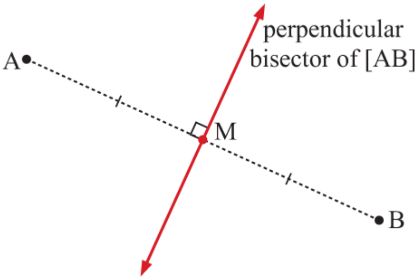
b Show that the point $R(\frac{11}{2},\ \frac{5}{2})$ lies on both tangents.

c Show that $PR = QR$.

F PERPENDICULAR BISECTORS

If A and B are two points, the **perpendicular bisector** of [AB] is the line perpendicular to [AB], passing through the midpoint of [AB].

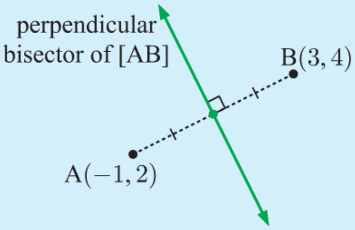
The perpendicular bisector of [AB] divides the number plane into two regions. On one side of the line are points that are closer to A than to B, and on the other side are points that are closer to B than to A.



Points on the perpendicular bisector of [AB] are **equidistant** from A and B.

Example 18 **Self Tutor**

Given $A(-1,\ 2)$ and $B(3,\ 4)$, find the equation of the perpendicular bisector of [AB].



M is $\left(\dfrac{-1+3}{2},\ \dfrac{2+4}{2}\right)$ or $(1,\ 3)$.

The gradient of [AB] is $\dfrac{4-2}{3--1} = \frac{2}{4} = \frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{2}{1}$

$\therefore$ the equation of the perpendicular bisector is $2x+y=2(1)+(3)$

which is $2x+y=5$.

EXERCISE 6F

1 Find the equation of the perpendicular bisector of [AB] for:

a A$(3, -3)$ and B$(1, -1)$

b A$(1, 3)$ and B$(-3, 5)$

c A$(3, 1)$ and B$(-3, 6)$

d A$(4, -2)$ and B$(4, 4)$.

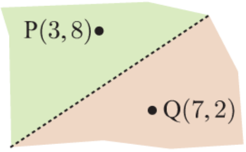
2 Suppose A is $(-1, -4)$ and B is $(3, 2)$.

a Find the equation of the perpendicular bisector of [AB].

b Show that C$(-5, 3)$ lies on the perpendicular bisector.

c Show that C is equidistant from A and B.

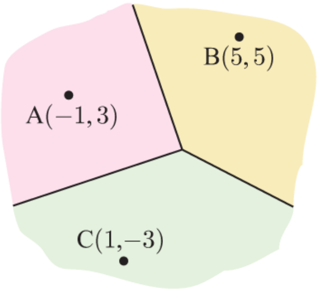
3 Two Post Offices are located at P$(3, 8)$ and Q$(7, 2)$ on a Council map. Find the equation of the line which should form the boundary between the two regions serviced by the Post Offices.



4 The **Voronoi** diagram alongside shows the location of three Post Offices and the corresponding regions of closest proximity. The Voronoi edges are the perpendicular bisectors of [AB], [BC], and [CA] respectively. Find:

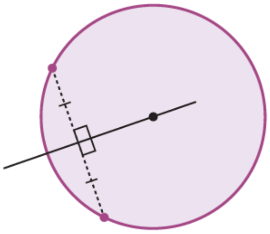
a the equations of the Voronoi edges

b the coordinates of the point where the Voronoi edges meet.



5 Consider the points A(x_1, y_1) and B(x_2, y_2). Show that the equation of the perpendicular bisector of [AB] is $(x_2 - x_1)x + (y_2 - y_1)y = \dfrac{({x_2}^2 + {y_2}^2) - ({x_1}^2 + {y_1}^2)}{2}$.

6



The perpendicular bisector of a chord of a circle, passes through its centre.

Find the centre of a circle passing through points P$(5, 7)$, Q$(7, 1)$, and R$(-1, 5)$.

Hint: Find the perpendicular bisectors of [PQ] and [QR], and solve them simultaneously.

7 Triangle ABC has the vertices shown.

a Find the coordinates of P, Q, and R, the midpoints of [AB], [BC], and [AC] respectively.

b Find the equation of the perpendicular bisector of:

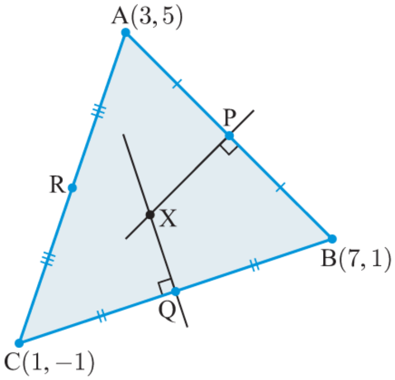
i [AB] **ii** [BC] **iii** [AC]

c Find the coordinates of X, the point of intersection of the perpendicular bisector of [AB] and the perpendicular bisector of [BC].

d Does X lie on the perpendicular bisector of [AC]?

e What does your result from **d** suggest about the perpendicular bisectors of the sides of a triangle?

f What is special about the point X in relation to the vertices of triangle ABC?



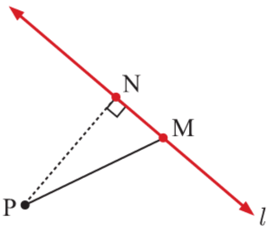
G DISTANCE FROM A POINT TO A LINE

When we talk about the distance from a point to a line, we actually mean the *shortest* distance from the point to the line.

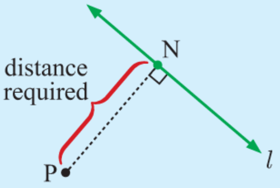
Suppose N is the foot of the perpendicular from P to the line l.

If M is any point on the line other than at N, then triangle MNP is right angled with hypotenuse [MP], and so $\text{MP} \geqslant \text{NP}$.

Hence NP is the shortest distance from P to line l.



The distance from a point P to a line l is the distance from P to N, where N is the point on l such that [NP] is perpendicular to l.



FINDING THE DISTANCE

To find the shortest distance from a point P to a line l we follow these steps:

Step 1: Find the gradient of the line l, and hence the gradient of [NP].
Step 2: Find the equation of the line segment [NP].
Step 3: Find the coordinates of N by solving simultaneously the equations of line l and line segment [NP].
Step 4: Find the distance NP using the distance formula.

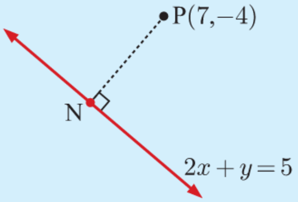
Example 19

Self Tutor

Find the distance from $\text{P}(7, -4)$ to the line with equation $2x + y = 5$.

Step 1: The gradient of $2x + y = 5$ is $-\frac{2}{1}$

$\therefore$ the gradient of [NP] is $\frac{1}{2}$



Step 2: The equation of [NP] is

$$x - 2y = (7) - 2(-4)$$

which is $x - 2y = 15$

Step 3: We now solve simultaneously: $\begin{cases} 2x + y = 5 & \text{.... (1)} \\ x - 2y = 15 & \text{.... (2)} \end{cases}$

$$\begin{aligned} 4x + 2y &= 10 \quad \{(1) \times 2\} \\ x - 2y &= 15 \quad \{(2)\} \\ \hline \text{Adding,} \quad 5x &= 25 \\ \therefore \quad x &= 5 \end{aligned}$$

$\therefore$ N is $(5, -5)$.

When $x = 5$, $2(5) + y = 5$

$\therefore$ $10 + y = 5$

$\therefore$ $y = -5$

Step 4: $\text{NP} = \sqrt{(7-5)^2 + (-4 - -5)^2}$
$= \sqrt{2^2 + 1^2}$
$= \sqrt{5}$ units

EXERCISE 6G

1 Find the distance from:

a $(7, -4)$ to $y = 3x - 5$
b $(-6, 0)$ to $y = 3 - 2x$
c $(8, -5)$ to $y = -2x - 4$
d $(-10, 9)$ to $y = -4x + 3$
e $(-2, 8)$ to $3x - y = 6$
f $(1, 7)$ to $4x - 3y = 8$.

2 Find the distance between the following pairs of parallel lines:

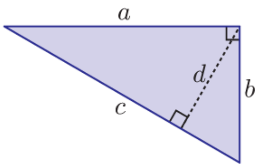
a $y = 3x + 2$ and $y = 3x - 8$
b $3x + 4y = 4$ and $3x + 4y = -16$

Hint: Find any point on one of the lines, then find the distance from this point to the other line.

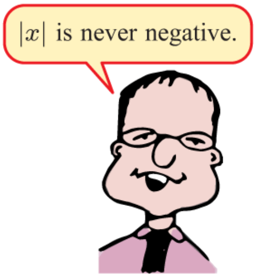
3 A straight water pipeline passes through two points with map references $(3, 2)$ and $(7, -1)$. The shortest spur pipe from the pipeline to the farm at $\text{P}(9, 7)$ is [NP].

a Find the coordinates of N.
b Find the length of the pipeline [NP] given that the grid reference scale is $1 \text{ unit} \equiv 0.5 \text{ km}$.

4 **a** For the diagram alongside, write *two* expressions for the area of the shaded triangle. Hence show that $d = \dfrac{ab}{\sqrt{a^2+b^2}}$.



b The **modulus** of x is $|x|$. It is the *size* of x, ignoring its sign, and can be defined by $|x| = \sqrt{x^2}$.
A property of modulus is that $|xy| = |x|\,|y|$ for all real numbers x, y.

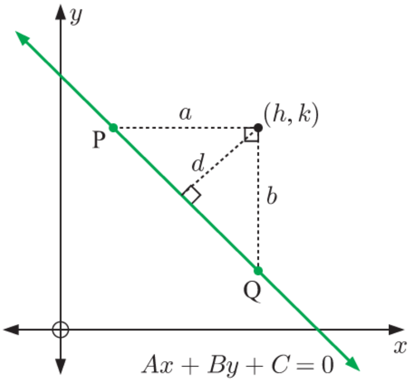


Consider the shortest distance d from a point (h, k) to the line $Ax + By + C = 0$.
Point P is the point on the line with y-coordinate k.
Point Q is the point on the line with x-coordinate h.
Show that:

i the distance $a = \dfrac{|Ah + Bk + C|}{|A|}$

ii the distance $b = \dfrac{|Ah + Bk + C|}{|B|}$

iii the distance $d = \dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$.



H 3-DIMENSIONAL COORDINATE GEOMETRY

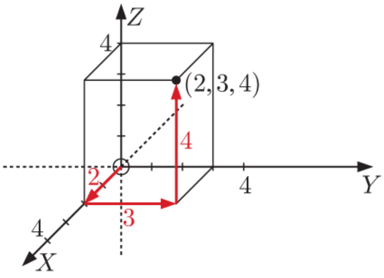
In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the X-axis, the Y-axis, and the Z-axis.

3D-POINT PLOTTER

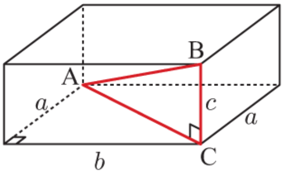
Any point in space can then be specified using an ordered triple in the form (x, y, z).

We generally suppose that the Y and Z-axes are in the plane of the page, and the X-axis is coming out of the page as shown.

The point $(2, 3, 4)$ is found by starting at the origin $O(0, 0, 0)$, moving 2 units along the X-axis, 3 units in the Y-direction, and then 4 units in the Z-direction.



We see that $(2, 3, 4)$ is located on the corner of a rectangular prism opposite O.



Now consider the rectangular prism illustrated, in which A is opposite B.

$$\begin{aligned} & AC^2 = a^2 + b^2 && \{\text{Pythagoras}\} \\ \text{and} \quad & AB^2 = AC^2 + c^2 && \{\text{Pythagoras}\} \\ \therefore \quad & AB^2 = a^2 + b^2 + c^2 && \\ \therefore \quad & AB = \sqrt{a^2 + b^2 + c^2} && \{AB > 0\} \end{aligned}$$

Suppose A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2).

- The **distance** $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The **midpoint** of [AB] is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

Example 20 **Self Tutor**

Consider $A(3, -1, 2)$ and $B(-1, 2, 4)$. Find:

a the distance AB **b** the midpoint of [AB].

a
$$\begin{aligned} AB &= \sqrt{(-1 - 3)^2 + (2 - -1)^2 + (4 - 2)^2} \\ &= \sqrt{(-4)^2 + 3^2 + 2^2} \\ &= \sqrt{16 + 9 + 4} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

b The midpoint is $\left(\dfrac{3 + -1}{2}, \dfrac{-1 + 2}{2}, \dfrac{2 + 4}{2}\right)$,

which is $(1, \frac{1}{2}, 3)$.

EXERCISE 6H

PRINTABLE 3-D PLOTTER PAPER

1 On separate axes, plot the points:

a $(4, 0, 0)$ **b** $(0, 2, 0)$ **c** $(0, 0, -3)$ **d** $(1, 2, 0)$
e $(2, 0, 4)$ **f** $(0, 3, -1)$ **g** $(2, 2, 2)$ **h** $(2, -1, 3)$
i $(4, 1, 2)$ **j** $(-2, 2, 3)$ **k** $(-1, 1, -1)$ **l** $(-3, 2, -1)$

2 For these pairs of points find:

i the distance AB **ii** the midpoint of [AB].

a $A(2, 3, -4)$ and $B(0, -1, 2)$ **b** $A(0, 0, 0)$ and $B(2, -4, 4)$
c $A(1, 1, 1)$ and $B(3, 3, 3)$ **d** $A(-1, 2, 4)$ and $B(4, -1, 3)$

3 Find the nature of triangle ABC given that:

a A is $(3, -3, 6)$, B is $(6, 2, 4)$, and C is $(4, -1, 3)$
b A is $(1, -2, 2)$, B is $(-8, 4, 17)$, and C is $(3, 6, 0)$.

4 Find k if the distance from $P(1, 2, 3)$ to $Q(k, 1, -1)$ is 6 units.

5 Find the relationship between x, y, and z if the point $P(x, y, z)$:

a is always 2 units from $O(0, 0, 0)$
b is always 4 units from $A(1, 2, 3)$.

Comment on your answer in each case.

6 Illustrate and describe these sets:

a $\{(x, y, z) \mid y = 2\}$ **b** $\{(x, y, z) \mid x = 1,\ y = 2\}$
c $\{(x, y, z) \mid x^2 + y^2 = 1,\ z = 0\}$ **d** $\{(x, y, z) \mid x^2 + y^2 + z^2 = 4\}$
e $\{(x, y, z) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2,\ z = 3\}$
f $\{(x, y, z) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 1\}$.

REVIEW SET 6A

1 Find the midpoint of the line segment joining $A(-2, 3)$ to $B(-4, 3)$.

2 Find the distance from $C(-3, -2)$ to $D(0, 5)$.

3 Find the gradient of all lines perpendicular to a line with gradient $\frac{2}{3}$.

4 $K(-3, 2)$ and $L(3, m)$ are $\sqrt{52}$ units apart. Find m.

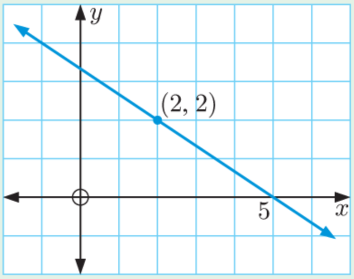
5 Find t given that the line joining $(-1, t)$ and $(5, -3)$ has gradient $\frac{4}{3}$.

6 Show that $A(1, -2)$, $B(4, 4)$, and $C(5, 6)$ are collinear.

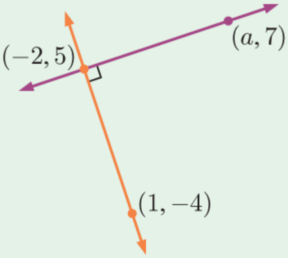
7 Find the equation of the line:

a with gradient -2 and y-intercept 7
b passing through $(-1, 3)$ and $(2, 1)$
c parallel to a line with gradient $\frac{3}{2}$, and passing through $(5, 0)$.

8 Find the equation of the line:



9 Find the value of a:

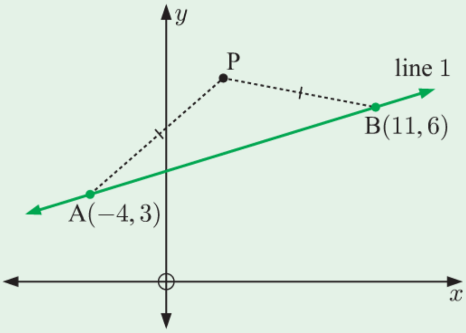


10 Consider the points $A(-3, 1)$, $B(1, 4)$, and $C(4, 0)$.

- **a** Show that triangle ABC is right angled and isosceles.
- **b** Find the midpoint X of [AC].
- **c** Use gradients to verify that [BX] is perpendicular to [AC].

11

- **a** Find, in general form, the equation of line 1.
- **b** Point P has x-coordinate 3, and is equidistant from A and B. Find the coordinates of P.
- **c** Find the equation of line 2, which is perpendicular to line 1, and passes through P.
- **d**
  - **i** Find the midpoint M of [AB].
  - **ii** Show that M lies on line 2.



12 Find the equation of the:

- **a** tangent to the circle with centre $(-1, 2)$ at the point $(3, 1)$
- **b** perpendicular bisector of [AB] for $A(2, 6)$ and $B(5, -2)$.

13 Find the shortest distance from $A(3, 5)$ to the line with equation $3x + 2y = 6$.

14 For $P(-1, 2, 3)$ and $Q(1, -2, -3)$, find:

- **a** the distance PQ
- **b** the midpoint of [PQ].

15 The distance between $P(1, 3, -1)$ and $Q(2, 1, k)$, is $\sqrt{30}$ units. Find k.

REVIEW SET 6B

1 Consider the points $S(7, -2)$ and $T(-1, 1)$.

- **a** Find the distance ST.
- **b** Determine the midpoint of [ST].

2 Find, in general form, the equation of the line passing through $P(-3, 2)$ and $Q(3, -1)$.

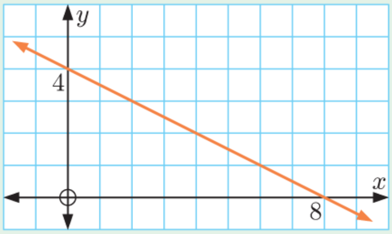
3

- **a** Find the gradient of all lines perpendicular to a line with gradient $-\frac{1}{2}$.
- **b** Determine whether the line $2x + y = 3$ is perpendicular to a line with gradient $-\frac{1}{2}$.

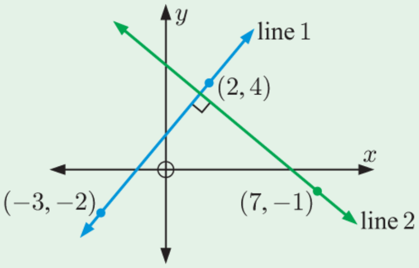
4 $X(-2, 3)$ and $Y(a, -1)$ are $\sqrt{17}$ units apart. Find the value of a.

5 Find b given that $A(-6, 2)$, $B(b, 0)$, and $C(3, -4)$ are collinear.

6 Determine the equation of the line:



7 Find the equation of line 2.



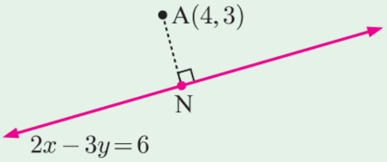
8 Find, in gradient-intercept form, the equation of the line passing through $(1, -2)$ and $(3, 4)$.

9 $A(-3, 2)$, $B(2, 3)$, $C(4, -1)$, and $D(-1, -2)$ are the vertices of quadrilateral ABCD.

- **a** **i** Find the gradient of each side of the quadrilateral.
  - **ii** What can you deduce about quadrilateral ABCD?
- **b** **i** Find the midpoints of the diagonals [AC] and [BD].
  - **ii** What property of parallelograms does this check?
- **c** **i** Find the gradients of the diagonals [AC] and [BD].
  - **ii** What does the product of these gradients tell us about quadrilateral ABCD?

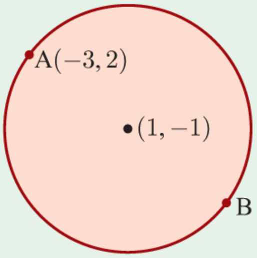
10 Find:

- **a** the coordinates of point N
- **b** the shortest distance from A to N.

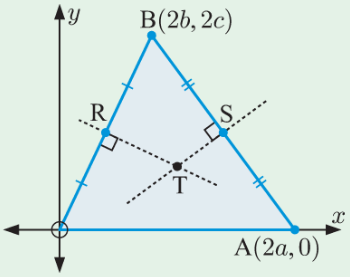


11 [AB] is a diameter of a circle with centre $(1, -1)$.
A has coordinates $(-3, 2)$.

- **a** Find the radius of the circle.
- **b** Find the equation of the tangent at A.
- **c** Find the coordinates of B.
- **d** Find the equation of the tangent at B.



12
- **a** Show that the perpendicular bisector of [OB] has equation $bx + cy = b^2 + c^2$.
- **b** Show that the perpendicular bisector of [AB] has equation $(a - b)\,x - cy = a^2 - b^2 - c^2$.
- **c** Prove that the perpendicular bisector of [OA] passes through the point of intersection of the other two perpendicular bisectors of $\triangle$OAB.



13 Find the distance between the parallel lines $2x + y = -5$ and $2x + y = 7$.

14 How far is $A(-1, -2, 5)$ from the origin O?

15 $P(x, y, z)$ is equidistant from $(-1, 1, 0)$ and $(2, 0, 0)$. Deduce that $y = 3x - 1$.

EXERCISE 6A

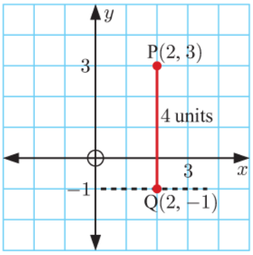
1 **a** $2\sqrt{2}$ units **b** 7 units **c** $2\sqrt{5}$ units **d** $3\sqrt{5}$ units
e 7 units **f** $\sqrt{5}$ units **g** $\sqrt{10}$ units **h** $\frac{\sqrt{5}}{2}$ units

2 **a** $10\sqrt{2} \approx 14.1$ km **b** $20\sqrt{5} \approx 44.7$ km
c $10\sqrt{26} \approx 51.0$ km

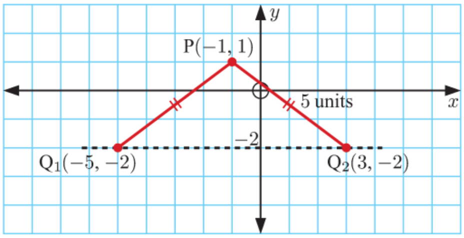
3 **a** isosceles with $AB = AC = \sqrt{85}$ units **b** scalene
c isosceles (and right angled at B) with $AB = BC$
d isosceles with $BC = AC = \sqrt{7}$ units
e equilateral, all sides $2\sqrt{3}$ units
f isosceles $(AC = BC)$ if $b \neq 2 \pm a\sqrt{3}$
equilateral if $b = 2 \pm a\sqrt{3}$

4 **a** right angled at B **b** not right angled
c right angled at A **d** not right angled

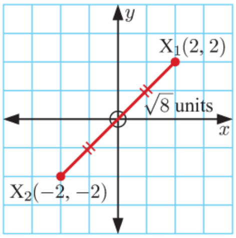
5 **a** $a = 2$



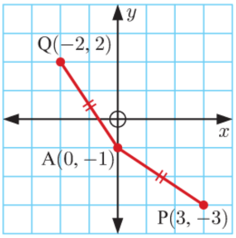
b $a = 3$ or -5



c $a = \pm 2$

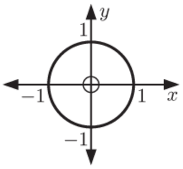


d $a = -1$



6 **a** **i** $x^2 + y^2 = 9$ **ii** $(x-1)^2 + (y-3)^2 = 4$

b



This set represents a circle with centre (0, 0) and radius 1 unit.

7 Let R be $(a, 0)$. Find PQ, PR, and RQ.
Solve $PQ = PR$, $PQ = RQ$, $PR = RQ$.
$(10, 0)$, $(-8, 0)$, $(-3, 0)$, $(-7, 0)$, and $(-\frac{101}{12}, 0)$

EXERCISE 6B

1 **a** $(5, 3)$ **b** $(1, -1)$ **c** $(\frac{3}{2}, 3)$ **d** $(0, 4)$
e $(2, -\frac{3}{2})$ **f** $(1, \frac{5}{2})$

2 **a** B(0, −6) **b** B(5, −2) **c** B(0, 6) **d** B(0, 7)
e B(−7, 3) **f** B(−3, 0)

3 C(1, −3) **4** P(7, −3) **5** $X(\frac{3}{2}, \frac{3}{2})$, S(−2, 0)

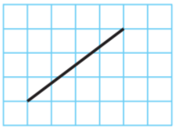
6 $\frac{\sqrt{89}}{2}$ units ≈ 4.72 units **7** $a = \frac{7}{3}$, $b = \frac{11}{2}$

8 **Hint:** Let the vertices be $A(a, b)$, $B(c, d)$, $C(e, f)$.
Use the midpoint formula to find six simple equations and solve them simultaneously.
Points are (3, −1), (7, 9), (9, 1).

EXERCISE 6C.1

1 **a** $\frac{1}{3}$ **b** 0 **c** -3 **d** $\frac{2}{3}$ **e** $-\frac{3}{4}$
f undefined **g** -4 **h** $-\frac{2}{5}$

2 **a**



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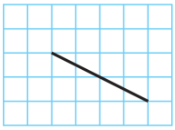

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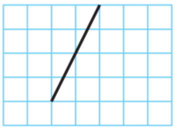

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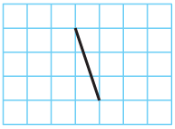

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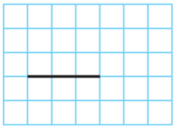

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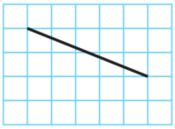

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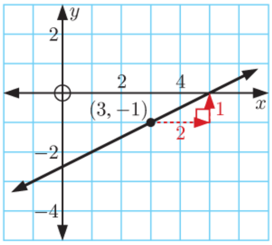

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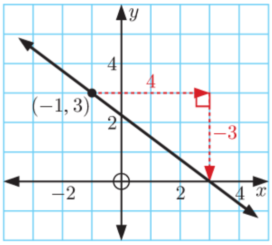

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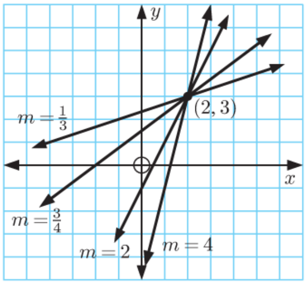

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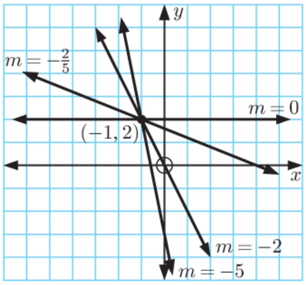
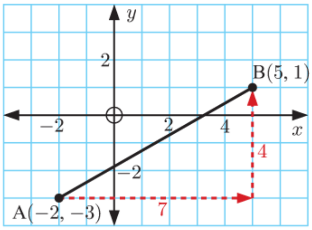


EXERCISE 6C.2

1 $\frac{4}{7}$



2 **a** $\frac{1}{5}$ **b** $\frac{1}{4}$ **c** 4 **d** 0 **e** undefined
f $\frac{2}{7}$ **g** $-\frac{2}{7}$ **h** 1 **i** $-\frac{4}{3}$

3 **a** $t = 19$ **b** $t = \frac{29}{2}$ **c** $t = \frac{13}{3}$ **d** $t = \frac{2}{3}$
e $t = 9$ **f** $t = 5$

4 $Q(0,\ 7)$, $R(\frac{28}{3},\ 0)$ **5** **c** gradient of [AC] $= 2$

EXERCISE 6D.1

1 **a** -2 **b** $-\frac{5}{2}$ **c** $-\frac{1}{3}$ **d** $-\frac{1}{7}$ **e** $\frac{5}{2}$ **f** $\frac{2}{7}$
g $\frac{3}{4}$ **h** 1

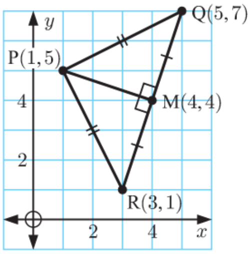
2 The line pairs in **c**, **d**, **f**, and **h** are perpendicular.

3 **a** [AB]: $\frac{3}{5}$, [BC]: -2, [CD]: $\frac{9}{2}$, [DE]: $-\frac{1}{2}$, [EF]: -2, [FA]: 2
b **i** [BC] $\parallel$ [EF] **ii** [DE] $\perp$ [FA]

4 $k = -2$ **5** **a** $t = 4$ **b** $t = 4$ **c** $t = 14$ **d** $t = \frac{22}{7}$

6 **a** $PQ = PR = 2\sqrt{5}$ units **b** $M(4,\ 4)$
c gradient of [PM] $= -\frac{1}{3}$,
gradient of [QR] $= 3$, and
their product is -1
$\therefore$ [PM] $\perp$ [QR]

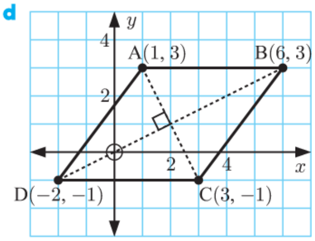
d



7 $M(0,\ 3)$ and $N(3,\ 3)$
a gradient of [MN] $= 0$, gradient of [AC] $= 0$
$\therefore$ [MN] $\parallel$ [AC]
b $MN = 3$ units and $AC = 6$ units $\therefore$ $MN = \frac{1}{2}(AC)$

8 **a** All sides have length 5 units. $\therefore$ ABCD is a rhombus.
b $(2,\ 1)$ and $(2,\ 1)$
c gradient of [AC] $= -2$,
gradient of [BD] $= \frac{1}{2}$
and their product $= -1$
$\therefore$ [AC] $\perp$ [BD]

d



9 **a** **i** $P(0,\ 5)$ **ii** $Q(\frac{9}{2},\ 2)$ **iii** $R(\frac{1}{2},\ -\frac{5}{2})$ **iv** $S(-4,\ \frac{1}{2})$
b **i** $-\frac{2}{3}$ **ii** $\frac{9}{8}$ **iii** $-\frac{2}{3}$ **iv** $\frac{9}{8}$
c PQRS is a parallelogram.

10 **a** $s = 6$ **b** **i** $\frac{1}{2}$ **ii** -2
c gradient of [PS] $\times$ gradient of [SQ] $= -1$
$\therefore$ $\widehat{PSQ} = 90°$

EXERCISE 6D.2

1 **a** gradient of [AB] $= \frac{4}{3}$, gradient of [BC] $= \frac{5}{4}$
$\therefore$ not collinear
b gradient of [PQ] $=$ gradient of [QR] $= \frac{6}{5}$
$\therefore$ P, Q, R are collinear
c gradient of [RS] $= -\frac{3}{11}$, gradient of [ST] $= -\frac{3}{2}$
$\therefore$ not collinear
d gradient of [AB] $=$ gradient of [BC] $= 3$
$\therefore$ A, B, C are collinear

2 **a** $c = 3$ **b** $c = -5$

3 **a** $M(1,\ -4)$
b gradient of [AM] $=$ gradient of [MC] $= 1$, $\therefore$ collinear
c gradient of [AC] $= 1$, gradient of [BD] $= -1$,
$\therefore$ perpendicular

EXERCISE 6E.1

1 **a** $y = 2x + 1$ **b** $y = -x + 1$ **c** $y = \frac{2}{3}x + 3$
d $y = -\frac{4}{5}x + \frac{6}{5}$ **e** $y = -\frac{3}{4}x - \frac{1}{2}$

2 **a** $4x - y = 7$ **b** $3x + 5y = -1$ **c** $x - 3y = -11$
d $3x + 4y = 24$ **e** $2x - 7y = 25$

3 **a** $y = x - 4$ **b** $y = -x + 4$ **c** $y = 6x + 16$
d $y = \frac{1}{2}x + 8$ **e** $y = -\frac{1}{3}x - \frac{7}{3}$ **f** $y = -\frac{3}{5}x - \frac{14}{5}$

4 **a** $3x - y = -1$ **b** $2x + y = 3$ **c** $x + 4y = 1$
d $5x - 3y = 10$ **e** $y = -3$ **f** $x = 2$

5 **a** $4x + 5y = 13$ **b** $4(12) + 5(-7) = 48 - 35 = 13$ ✓

6 **a** $2x - 5y = 10$ **b** $y = -2x - 2$ **c** $y = 2x + 6$
d $4x + 3y = 20$ **e** $x - 2y = -8$

7 **a** $-\frac{3}{4}$ **b** $4x - 3y = 10$ **8** $7x - 5y = -42$

9 $y = \frac{3}{2}x - 3$ **10** **a** $y = \frac{2}{3}x - \frac{5}{3}$ **b** $-\frac{5}{3}$

11 $x + 3y = 13$

12 **a** **i** $y = -\frac{1}{2}x + 2$ **ii** $y = 2x - 3$ **iii** $y = -\frac{1}{2}x + \frac{19}{2}$
b **Hint:** Solve $2x - 3 = -\frac{1}{2}x + \frac{19}{2}$.

13 **Hint:** P is $(a + b,\ c)$, Q is $(b,\ c)$
a gradient of [OP] $= \dfrac{c}{a + b}$ etc.
b gradient of [AQ] $= \dfrac{c}{b - 2a}$ etc.
c Solve the equations in **a** and **b** simultaneously.

EXERCISE 6E.2

1 **a** $x - 2y = 2$ **b** $2x - 3y = -19$ **c** $3x - 4y = 15$
d $3x - y = 11$ **e** $x + 3y = 13$ **f** $3x + 4y = -6$
g $2x + y = 4$ **h** $3x + y = 4$

2 **a** $-\frac{2}{3}$ **b** $\frac{3}{7}$ **c** $\frac{6}{11}$ **d** $-\frac{5}{6}$ **e** $-\frac{1}{2}$ **f** 3

3 **a** Parallel lines have the same gradient of $-\frac{3}{5}$.
$\therefore$ equations have the form $3x + 5y =$ a constant.
b $3x + 5y = 2$ has gradient $-\frac{3}{5}$
$\therefore$ perpendicular lines have gradient $\frac{5}{3}$.
$\therefore$ equations have the form $5x - 3y =$ a constant.

4 **a** $3x + 4y = 10$ **b** $2x - 5y = 3$ **c** $3x + y = -12$
d $x - 3y = 0$

5 **a** $\frac{2}{3}$ and $-\frac{6}{k}$ **b** $k = -9$ **c** $k = 4$

6 **a** $2\sqrt{34}$ km ≈ 11.7 km **b** $(6,\ -\frac{1}{2})$ **c** no
d **i** $11x + 8y = 31$
ii No, as $11(2) + 8(1) = 30 \neq 31$.

7 **a** $x - 3y = -16$ **b** $3x - 2y = 13$ **c** $2x - y = -3$
d $x = 5$

8 **a** **i** $x - 7y = -12$ **ii** $x + y = 8$
b $(\frac{11}{2}) - 7(\frac{5}{2}) = -\frac{24}{2} = -12$ ✓ $(\frac{11}{2}) + (\frac{5}{2}) = 8$ ✓
c $PR = QR = \frac{5\sqrt{2}}{2}$ units

EXERCISE 6F

1 **a** $x - y = 4$ **b** $2x - y = -6$
c $12x - 10y = -35$ **d** $y = 1$

2 **a** $2x + 3y = -1$ **b** $2(-5) + 3(3) = -1$ ✓
c $AC = BC = \sqrt{65}$ units

3 $2x - 3y = -5$

4 **a** $x + 2y = 5, \quad 3x + y = 10, \quad x - 3y = 0$ **b** $(3, 1)$

5 **Hint:** Start by finding the gradient and midpoint of [AB].

6 The perpendicular bisectors of [PQ] and [QR] are $x - 3y = -6$ and $2x - y = 3$. They meet at $(3, 3)$.

7 **a** P(5, 3), Q(4, 0), R(2, 2)
b **i** $x - y = 2$ **ii** $3x + y = 12$ **iii** $x + 3y = 8$
c $X(3\frac{1}{2}, 1\frac{1}{2})$ **d** Yes, $(\frac{7}{2}) + 3(\frac{3}{2}) = 8$ ✓
e The perpendicular bisectors meet at a point.
f X is the centre of the circle which could be drawn through A, B, and C.

EXERCISE 6G

1 **a** $2\sqrt{10}$ units **b** $3\sqrt{5}$ units **c** $3\sqrt{5}$ units
d $2\sqrt{17}$ units **e** $2\sqrt{10}$ units **f** 5 units

2 **a** $\sqrt{10}$ units **b** 4 units

3 **a** N(4.44, 0.92) **b** 3.8 km

4 **a** $A = \frac{ab}{2}, \quad A = \frac{cd}{2} = \frac{d\sqrt{a^2+b^2}}{2}$

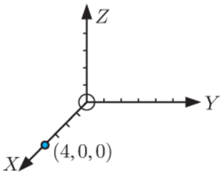
b **i** When $y = k, \quad Ax + Bk + C = 0$

$$\therefore \ x = \frac{-Bk - C}{A}$$

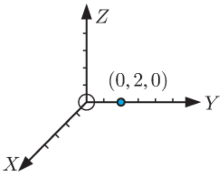
$$\therefore \ \text{P is } \left(\frac{-Bk - C}{A}, k\right) \text{ etc.}$$

EXERCISE 6H

1 **a**



b



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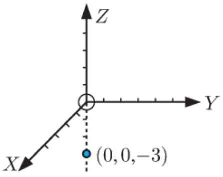

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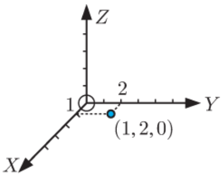

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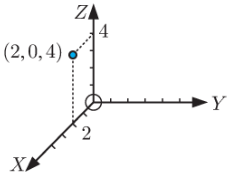

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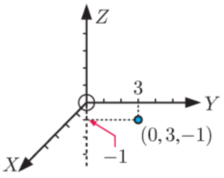

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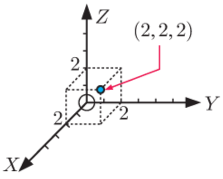

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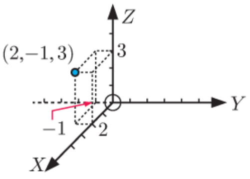

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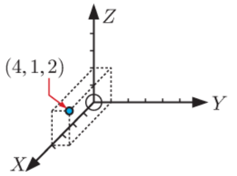

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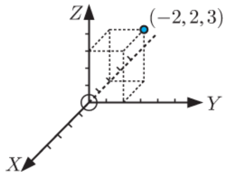

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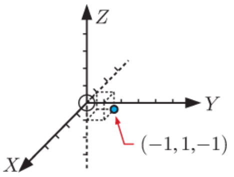

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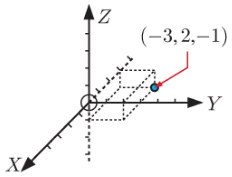


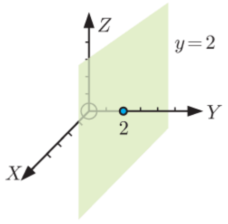
2 **a** **i** $2\sqrt{14}$ units **ii** M(1, 1, −1)
b **i** 6 units **ii** M(1, −2, 2)
c **i** $2\sqrt{3}$ units **ii** M(2, 2, 2)
d **i** $\sqrt{35}$ units **ii** $M(\frac{3}{2}, \frac{1}{2}, \frac{7}{2})$

3 **a** Isosceles with $AC = BC = \sqrt{14}$ units.
b △ABC is right angled at A.

4 $k = 1 \pm \sqrt{19}$

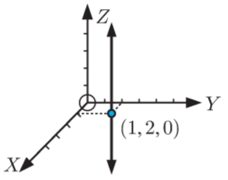
5 **a** $x^2 + y^2 + z^2 = 4$ and must be the equation of a sphere, centre (0, 0, 0), radius 2 units.
b $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$ which is the equation of a sphere, centre (1, 2, 3), radius 4 units.

6 **a**



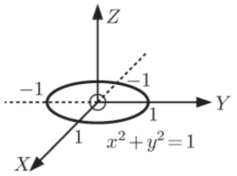
A plane parallel to the XOZ plane passing through (0, 2, 0).

b



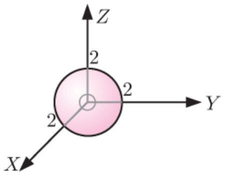
A line parallel to the Z-axis passing through (1, 2, 0).

c



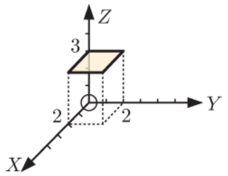
A circle in the XOY plane, centre (0, 0, 0), radius 1 unit.

d



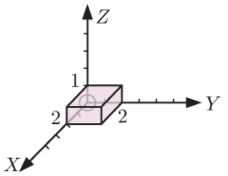
A sphere, centre (0, 0, 0), radius 2 units.

e



A 2 by 2 square plane 3 units above the XOY plane (as shown).

f



All points on and within a $2 \times 2 \times 1$ rectangular prism (as shown).

REVIEW SET 6A

1 (−3, 3) **2** $\sqrt{58}$ units **3** $-\frac{3}{2}$ **4** $m = 6$ or -2

5 $t = -11$

6 Gradient of [AB] and [BC] $= 2$.
$\therefore$ [AB] $\parallel$ [BC] with B common.

7 **a** $y = -2x + 7$ **b** $2x + 3y = 7$ **c** $3x - 2y = 15$

8 $2x + 3y = 10$ **9** $a = 4$

10 **a** AB $=$ BC $= 5$ units, $m_{\text{AB}} = \frac{3}{4}$, $m_{\text{BC}} = -\frac{4}{3}$
$\therefore$ right angle at B.
b X$(\frac{1}{2}, \frac{1}{2})$
c gradient of [BX] $= 7$, gradient of [AC] $= -\frac{1}{7}$
and $7 \times -\frac{1}{7} = -1$, $\therefore$ [BX] $\perp$ [AC].

11 **a** $x - 5y = -19$ **b** P$(3, 7)$ **c** $5x + y = 22$
d **i** $(\frac{7}{2}, \frac{9}{2})$ **ii** $5(\frac{7}{2}) + \frac{9}{2} = \frac{44}{2} = 22$ ✓

12 **a** $y = 4x - 11$ **b** $6x - 16y = -11$ **13** $\sqrt{13}$ units

14 **a** $2\sqrt{14}$ units **b** M$(0, 0, 0)$ **15** $k = 4$ or -6

REVIEW SET 6B

1 **a** ST $= \sqrt{73}$ units **b** $(3, -\frac{1}{2})$ **2** $x + 2y = 1$

3 **a** 2 **b** No, as $2x + y = 3$ has gradient -2.

4 $a = -1$ or -3 **5** $b = -3$

6 $x + 2y = 8$ **7** $5x + 6y = 29$ **8** $y = 3x - 5$

9 **a** **i** [AB]: $\frac{1}{5}$, [BC]: -2, [CD]: $\frac{1}{5}$, [AD]: -2
ii ABCD is a parallelogram.
b **i** $(\frac{1}{2}, \frac{1}{2})$ for both diagonals.
ii The diagonals of a parallelogram bisect each other.
c **i** [AC]: $-\frac{3}{7}$, [BD]: $\frac{5}{3}$
ii product $= -\frac{5}{7}$ $\therefore$ not a rhombus

10 **a** N$(\frac{66}{13}, \frac{18}{13})$ **b** $\frac{7}{\sqrt{13}}$ units

11 **a** 5 units **b** $4x - 3y = -18$ **c** B$(5, -4)$
d $4x - 3y = 32$

12 **a** R is (b, c), [OB] has gradient $\frac{c}{b}$, etc.
c **Hint:** Show that the x-coordinate of T is a.

13 distance $= \frac{12}{\sqrt{5}}$ units **14** $\sqrt{30}$ units

EXERCISE 7A

1 **a** $\triangle$ABC $\cong$ $\triangle$FED {AAcorS}
b $\triangle$PQR $\cong$ $\triangle$ZYX {SAS}
c $\triangle$ABC $\cong$ $\triangle$EDF {AAcorS}
d $\triangle$ABC $\cong$ $\triangle$LKM {SSS}
e $\triangle$XYZ $\cong$ $\triangle$FED {RHS} **f** not congruent
g not enough information **h** not enough information
i $\triangle$ABC $\cong$ $\triangle$PQR {SSS} **j** $\triangle$ABC $\cong$ $\triangle$FED {AAcorS}

2 **a** **B** and **D** {SAS} **b** **A** and **D** {RHS}
c **B** and **C** {AAcorS} **d** **A** and **D** {SSS}

EXERCISE 7B

1 $\triangle$ABC $\cong$ $\triangle$EDC {AAcorS}

2 **a** $\triangle$ABD $\cong$ $\triangle$CBD {SSS} **b** **i** 47° **ii** 51°

3 **a** Join [AC].
Hint: Show $\triangle$ABC $\cong$ $\triangle$CDA {AAcorS}
b Now join [DB] and let the diagonals meet at M.
Hint: Show $\triangle$AMD $\cong$ $\triangle$CMB {AAcorS}

4 **a** **Hint:** Show $\triangle$XYZ $\cong$ $\triangle$ZWX {SSS}
b $\text{W}\widehat{\text{Z}}\text{X} = \text{Y}\widehat{\text{X}}\text{Z}$ so WZ $\parallel$ XY
and $\text{W}\widehat{\text{X}}\text{Z} = \text{Y}\widehat{\text{Z}}\text{X}$ so WX $\parallel$ ZY

5 **Hint:** Join [OA], [OB], and [OP], and
show $\triangle$OAP $\cong$ $\triangle$OBP. {RHS}

6 **Hint:** Join [AC], [BC], and [CX], and
show $\triangle$ACX $\cong$ $\triangle$BCX. {RHS}

7 **Hint:** Show $\triangle$APB $\cong$ $\triangle$AQC {AAcorS}

8 **Hint:** Join [AX], [BX], and [CX].
Show $\triangle$AQX $\cong$ $\triangle$BQX {SAS}
and $\triangle$APX $\cong$ $\triangle$CPX {SAS}

9 **Hint:** Join [WZ] and [ZY].
Show $\triangle$WAZ $\cong$ $\triangle$YDZ {SAS}

EXERCISE 7C

1 **e** **Hint:** Let $\text{N}\widehat{\text{J}}\text{M}$ be $\alpha°$; find all other angles in terms of α.
f **Hint:** Let $\text{A}\widehat{\text{C}}\text{B} = \alpha$, then $\text{A}\widehat{\text{B}}\text{C} = \alpha$ and $\text{B}\widehat{\text{D}}\text{C} = \alpha$.

2 **a** $x = 1.2$ **b** $x = \frac{10}{3}$ **c** $x = \frac{20}{7}$ **d** $x = 7.5$
e $x = 10.8$ **f** $x = \frac{8}{3}$ **g** $x = \frac{4}{3}$ **h** $x = 4.8$
i $x = \frac{35}{3}$

3 7 m **4** ≈ 1.62 m **5** 10.625 km

6 ≈ 651 m **7** 8 cm

8 She is correct as the viewing region is always 20 m long no matter where the camera is located on [AB].

9 1.52 m **10** ≈ 35.7 m **11** ≈ 5.56 cm

EXERCISE 7D

1 **a** $x \approx 14.2$ **b** $x \approx 30.7$ **c** $x \approx 41.5$
d $x \approx 6.26$ **e** $x = 8$ **f** $x \approx 10.7$

2 **a** $x = 96$ **b** $x = 432$ **c** $x \approx 0.150$
d $x = 7.5$ **e** $x = 13.2$ **f** $x \approx 6.94$

3 **a** 12.544 cm^2 **b** 6.144 cm^2

4 **a** $x = 1.5$ **b** 17.6 m^2 **5** 30 cm

6 **a** **i** It is multiplied by 8. **ii** It is increased by 72.8%.
b **i** It is divided by 8. **ii** It is increased by 237.5%.
c **i** ≈ 48.8 cm^3 **ii** 1310.72 grams

7 Length [RS] is the length of [PM] enlarged with scale factor $k = 2$.
$\therefore$ the area of the similar triangle will be enlarged by a factor of $k^2 = 4$.

8 **a** similar **b** not similar

9 **a** 10 cm **b** 30 cm^2 **c** 6 mm, 2.25 mm **d** 1.25 mL

10 **B** and **F**
Hint: Create a table for each pair of glasses.

	k	k^3	*Calc. V*	*Actual V*
A, B	$\frac{10}{8.5}$	≈ 1.628	≈ 204	160

✗

REVIEW SET 7A

1 **a** **B** and **C** {AAcorS} **b** **A** and **C** {AAcorS}

4 **a** $x = \frac{13}{8}$ **b** $x = \frac{23}{3}$ **c** $x = \frac{40}{3}$

5 **a** 38.4 cm^2 **b** 28.8 cm^2

6 **Hint:** Show $\triangle$s APQ and ABC are similar, then $\triangle$s PBC and QCB congruent, etc.

7 ≈ 117 m