Differential Calculus revision 2 *[163 marks]*

A curve has equation $3x - 2y^2\mathrm{e}^{x-1} = 2$.

1a. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.   *[5 marks]*

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1b. Find the equations of the tangents to this curve at the points where the   *[4 marks]*
curve intersects the line $x = 1$.

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Consider the curves C_1 and C_2 defined as follows

$C_1 : xy = 4$, $x > 0$

$C_2 : y^2 - x^2 = 2$, $x > 0$

2a. Using implicit differentiation, or otherwise, find $\frac{dy}{dx}$ for each curve in terms of x and y.   [4 marks]

2b. Let P(a, b) be the unique point where the curves C_1 and C_2 intersect.   [2 marks]
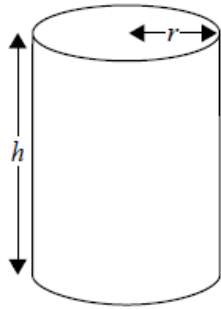Show that the tangent to C_1 at P is perpendicular to the tangent to C_2 at P.

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20π cm³.

diagram not to scale

3a. Express *h* in terms of *r*.   *[2 marks]*

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The material for the base and top of the can costs 10 cents per cm² and the material for the curved side costs 8 cents per cm². The total cost of the material, in cents, is *C*.

3b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. [4 marks]

3c. Given that there is a minimum value for *C*, find this minimum value in terms of π. *[9 marks]*

Let $f(x) = \frac{2-3x^5}{2x^3}$, $x \in \mathbb{R}$, $x \neq 0$.

4a. The graph of $y = f(x)$ has a local maximum at A. Find the coordinates   [5 marks]
of A.

4b. Show that there is exactly one point of inflexion, B, on the graph of   [5 marks]
$y = f(x)$.

4c. The coordinates of B can be expressed in the form $B\left(2^a,\ b \times 2^{-3a}\right)$ *[3 marks]* where a, $b \in \mathbb{Q}$. Find the value of a and the value of b.

4d. Sketch the graph of $y = f(x)$ showing clearly the position of the points   *[4 marks]* A and B.

Let $f(x) = x^2 - x$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.
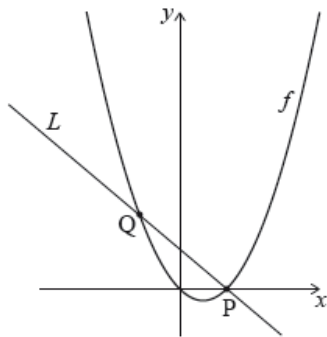
diagram not to scale
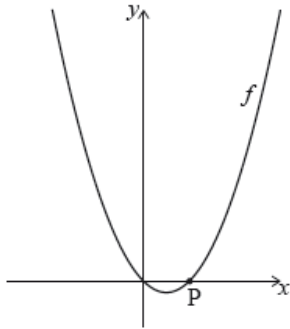
The graph of f crosses the x-axis at the origin and at the point $P(1, 0)$.

The line L intersects the graph of f at another point Q, as shown in the following diagram.

diagram not to scale

5.   Find the area of the region enclosed by the graph of f and the line L.   *[6 marks]*

6.   The folium of Descartes is a curve defined by the equation   *[8 marks]*

$x^3 + y^3 - 3xy = 0$, shown in the following diagram.

Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the y-axis.

Let $g(x) = p^x + q$, for $x, p, q \in \mathbb{R}, p > 1$. The point $A(0, a)$ lies on the graph of g.

Let $f(x) = g^{-1}(x)$. The point B lies on the graph of f and is the reflection of point A in the line $y = x$.

7a. Write down the coordinates of B. *[2 marks]*

The line L_1 is tangent to the graph of f at B.

7b. Given that $f'(a) = \frac{1}{\ln p}$, find the equation of L_1 **in terms of** x, p and q. *[5 marks]*

7c. The line L_2 is tangent to the graph of g at A and has equation   *[7 marks]*
$y = (\ln p)\, x + q + 1.$

The line L_2 passes through the point $(-2, \ -2)$.

The gradient of the normal to g at A is $\dfrac{1}{\ln\left(\frac{1}{3}\right)}$.

Find the equation of L_1 in terms of x.

A small cuboid box has a rectangular base of length $3x$ cm and width x cm, where $x > 0$. The height is y cm, where $y > 0$.

diagram not to scale
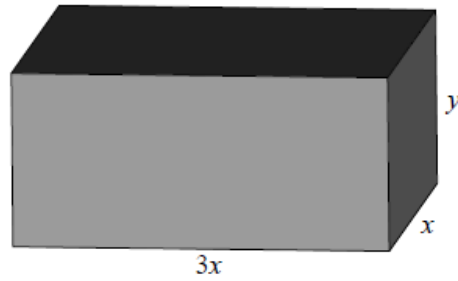
y

x

$3x$

The sum of the length, width and height is 12 cm.

8a. Write down an expression for y in terms of x.   *[1 mark]*

8b. Find an expression for V in terms of x.   *[2 marks]*

The volume of the box is V cm^3.

8c. Find $\frac{\mathrm{d}V}{\mathrm{d}x}$.   [2 marks]

8d. Find the value of x for which V is a maximum.   [4 marks]

8e. Justify your answer.   [3 marks]

8f. Find the maximum volume.   [2 marks]

9. A camera at point C is 3 m from the edge of a straight section of road as *[6 marks]*
    shown in the following diagram. The camera detects a car travelling
    along the road at $t = 0$. It then rotates, always pointing at the car, until the car
    passes O, the point on the edge of the road closest to the camera.
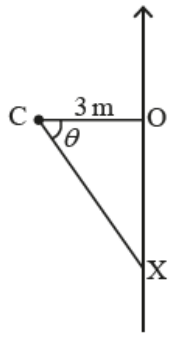
A car travels along the road at a speed of $24\,\text{ms}^{-1}$. Let the position of the car be X and let $O\hat{C}X = \theta$.

Find $\frac{d\theta}{dt}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

10. Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{dy}{dx} = 0$. [9 marks]

The curve C is given by the equation $y = x \tan\left(\frac{\pi xy}{4}\right)$.

11a. At the point (1, 1) , show that $\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$. [5 marks]

11b. Hence find the equation of the normal to C at the point (1, 1). [2 marks]

A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.

12a. Show that the volume of the cone may be expressed by $V = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$. [4 marks]

12b. Given that there is one inscribed cone having a maximum volume,   *[4 marks]*
show that the volume of this cone is $\frac{32\pi R^3}{81}$.

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Let $f(x) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.
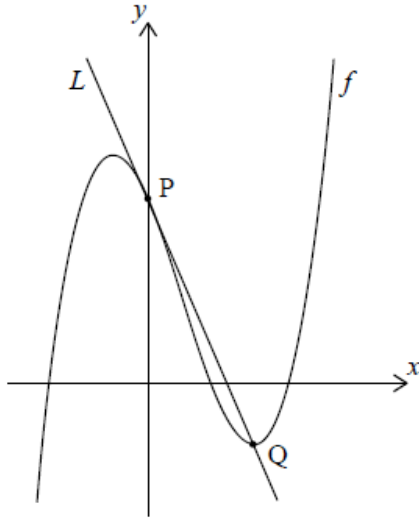
The graph of f crosses the y-axis at the point P. The line L is tangent to the graph of f at P.

13a. Find $f'(x)$.   [2 marks]

13b. Hence, find the equation of L in terms of a.   [4 marks]

13c. The graph of f has a local minimum at the point Q. The line L passes   [8 marks]
through Q.

Find the value of a.

14. Use l'Hôpital's rule to find $\lim\limits_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$.

[5 marks]

Consider the curve C defined by $y^2 = \sin(xy)$, $y \neq 0$.

15a. Show that $\frac{dy}{dx} = \frac{y\cos(xy)}{2y - x\cos(xy)}$.

[5 marks]

15b. Prove that, when $\frac{dy}{dx} = 0$, $y = \pm 1$. *[5 marks]*

15c. Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where   *[5 marks]*
$\frac{dy}{dx} = 0$.

16. Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point where $x = 0$. **[5 marks]**

17. Use l'Hôpital's rule to determine the value of $\lim\limits_{x \to 0}\left(\dfrac{2x\cos\left(x^2\right)}{5\tan x}\right)$. [5 marks]