Differential Calculus revision 2 *[163 marks]*

A curve has equation $3x - 2y^2\mathrm{e}^{x-1} = 2$.

1a. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to differentiate implicitly ***M1***

$3 - \left(4y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\right)\mathrm{e}^{x-1} = 0$ ***A1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\bullet\mathrm{e}^{1-x} - 2y^2}{4y}$ ***A1***

Note: This final answer may be expressed in a number of different ways.

[5 marks]

1b. Find the equations of the tangents to this curve at the points where the curve intersects the line $x = 1$. *[4 marks]*

Markscheme

$3 - 2y^2 = 2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3-2\bullet\frac{1}{2}}{\pm 4\sqrt{\frac{1}{2}}} = \pm\frac{\sqrt{2}}{2}$ ***M1***

at $\left(1,\ \sqrt{\frac{1}{2}}\right)$ the tangent is $y - \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}(x-1)$ and ***A1***

at $\left(1,\ -\sqrt{\frac{1}{2}}\right)$ the tangent is $y + \sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}(x-1)$ ***A1***

Note: These equations simplify to $y = \pm\frac{\sqrt{2}}{2}x$.

Note: Award ***A0M1A1A0*** if just the positive value of y is considered and just one tangent is found.

[4 marks]

Consider the curves C_1 and C_2 defined as follows

$C_1 : xy = 4$, $x > 0$

$C_2 : y^2 - x^2 = 2$, $x > 0$

2a. Using implicit differentiation, or otherwise, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each curve in terms of x and y. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$C_1 : y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)***

Note: ***M1*** is for use of both product rule and implicit differentiation.

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$ ***A1***

Note: Accept $-\frac{4}{x^2}$

$C_2 : 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$ ***A1***

Note: Accept $\pm\frac{x}{\sqrt{2+x^2}}$

[4 marks]

2b. Let P(a, b) be the unique point where the curves C_1 and C_2 intersect. *[2 marks]*

Show that the tangent to C_1 at P is perpendicular to the tangent to C_2 at P.

Markscheme

substituting a and b for x and y ***M1***

product of gradients at P is $\left(-\frac{b}{a}\right)\left(\frac{a}{b}\right) = -1$ or equivalent reasoning ***R1***

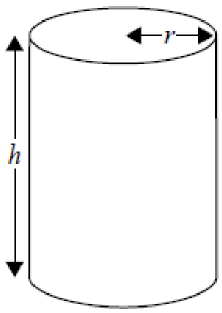
Note: The ***R1*** is dependent on the previous ***M1***.

so tangents are perpendicular ***AG***

[2 marks]

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20π cm^3.

diagram not to scale



3a. Express *h* in terms of *r*. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

correct equation for volume ***(A1)***

eg $\pi r^2 h = 20\pi$

$h = \frac{20}{r^2}$ ***A1 N2***

[2 marks]

The material for the base and top of the can costs 10 cents per cm^2 and the material for the curved side costs 8 cents per cm^2. The total cost of the material, in cents, is *C*.

3b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. *[4 marks]*

Markscheme

attempt to find formula for cost of parts ***(M1)***
eg 10 × two circles, 8 × curved side

correct expression for cost of two circles in terms of *r* (seen anywhere) ***A1***
eg $2\pi r^2 \times 10$

correct expression for cost of curved side (seen anywhere) ***(A1)***
eg $2\pi r \times h \times 8$

correct expression for cost of curved side in terms of *r* ***A1***
eg $8 \times 2\pi r \times \frac{20}{r^2},\ \frac{320\pi}{r^2}$

$C = 20\pi r^2 + \frac{320\pi}{r}$ ***AG N0***

[4 marks]

3c. Given that there is a minimum value for *C*, find this minimum value in terms of π. *[9 marks]*

Markscheme

recognize $C' = 0$ at minimum ***(R1)***
eg $C' = 0,\ \frac{\mathrm{d}C}{\mathrm{d}r} = 0$

correct differentiation (may be seen in equation)

$C' = 40\pi r - \frac{320\pi}{r^2}$ ***A1A1***

correct equation ***A1***
eg $40\pi r - \frac{320\pi}{r^2} = 0,\ 40\pi r \frac{320\pi}{r^2}$

correct working ***(A1)***
eg $40r^3 = 320,\ r^3 = 8$

r = 2 (m) ***A1***

attempt to substitute **their** value of *r* into *C*
eg $20\pi \times 4 + 320 \times \frac{\pi}{2}$ ***(M1)***

correct working
eg $80\pi + 160\pi$ ***(A1)***

240π (cents) ***A1 N3***

Note: Do not accept 753.6, 753.98 or 754, even if 240π is seen.

[9 marks]

Let $f\left(x\right)=\frac{2-3x^5}{2x^3},\ x\in\mathbb{R},\ x\neq 0.$

4a. The graph of $y=f\left(x\right)$ has a local maximum at A. Find the coordinates of A. *[5 marks]*

Markscheme

attempt to differentiate ***(M1)***

$f'\left(x\right)=-3x^{-4}-3x$ ***A1***

Note: Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example

$f'\left(x\right)=\frac{-15x^4\times 2x^3-6x^2\left(2-3x^5\right)}{\left(2x^3\right)^2}$.

$-\frac{3}{x^4}-3x=0$ ***M1***

$\Rightarrow x^5=-1\Rightarrow x=-1$ ***A1***

$\mathrm{A}\left(-1,\ -\frac{5}{2}\right)$ ***A1***

[5 marks]

4b. Show that there is exactly one point of inflexion, B, on the graph of $y=f\left(x\right)$. *[5 marks]*

Markscheme

$f''\left(x\right)=0$ ***M1***

$f''\left(x\right)=12x^{-5}-3\left(=0\right)$ ***A1***

Note: Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x=\sqrt[5]{4}\ \left(=2^{\frac{2}{5}}\right)$ ***A1***

hence (at most) one point of inflexion ***R1***

Note: This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''\left(x\right)$ changes sign at $x=\sqrt[5]{4}\ \left(=2^{\frac{2}{5}}\right)$ ***R1***

so exactly one point of inflexion

[5 marks]

4c. The coordinates of B can be expressed in the form B$\left(2^{a},\, b\times 2^{-3a}\right)$ where *a*, *b*$\in\mathbb{Q}$. Find the value of *a* and the value of *b*. *[3 marks]*

Markscheme

$x=\sqrt[5]{4}\ \ =2^{\frac{2}{5}}\left(\Rightarrow a=\frac{2}{5}\right)$ ***A1***

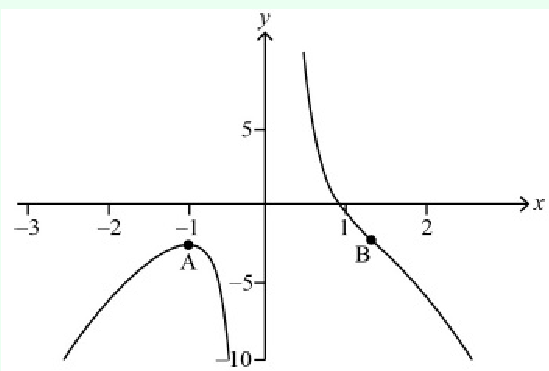
$f\left(2^{\frac{2}{5}}\right)=\frac{2-3\times 2^{2}}{2\times 2^{\frac{6}{5}}}=-5\times 2^{-\frac{6}{5}}\left(\Rightarrow b=-5\right)$ ***(M1)A1***

Note: Award ***M1*** for the substitution of their value for x into $f\left(x\right)$.

[3 marks]

4d. Sketch the graph of $y=f\left(x\right)$ showing clearly the position of the points A and B. *[4 marks]*

Markscheme



A1A1A1A1

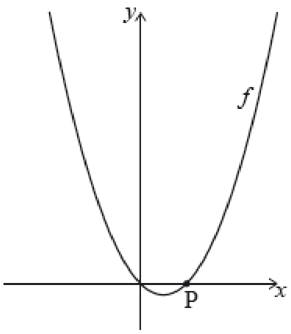
A1 for shape for *x* < 0
A1 for shape for *x* > 0
A1 for maximum at A
A1 for POI at B.

Note: Only award last two ***A1***s if A and B are placed in the correct quadrants, allowing for follow through.

[4 marks]

Let $f(x) = x^2 - x$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.

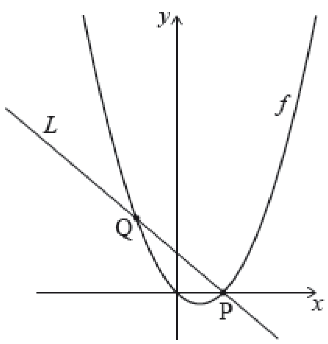
diagram not to scale



The graph of f crosses the x-axis at the origin and at the point $\mathrm{P}(1, 0)$.

The line L intersects the graph of f at another point Q, as shown in the following diagram.

diagram not to scale



5. Find the area of the region enclosed by the graph of f and the line L. *[6 marks]*

Markscheme

valid approach ***(M1)***

eg $\int L - f,\ \int_{-1}^{1}(1 - x^2)\mathrm{d}x$, splitting area into triangles and integrals

correct integration ***(A1)(A1)***

eg $\left[x - \frac{x^3}{3}\right]_{-1}^{1},\ -\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^2}{2} + x$

substituting **their** limits into **their** integrated function and subtracting (in any order) ***(M1)***

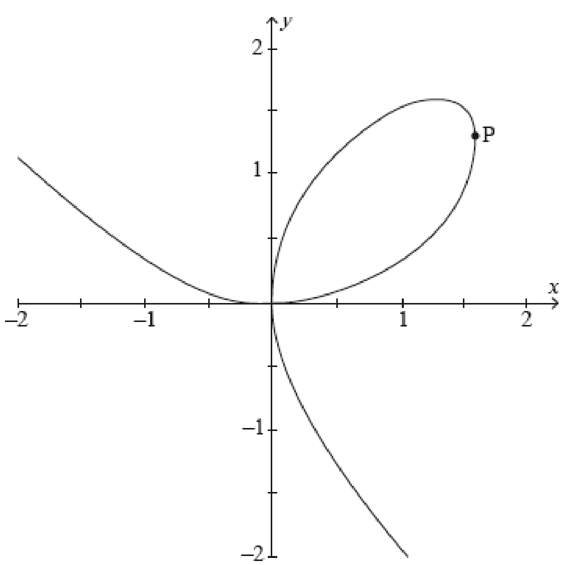
eg $1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)$

Note: Award ***M0*** for substituting into original or differentiated function.

area $= \frac{4}{3}$ ***A2*** ***N3***

[6 marks]

6. The folium of Descartes is a curve defined by the equation $x^3 + y^3 - 3xy = 0$, shown in the following diagram. *[8 marks]*



Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the y-axis.

Markscheme

* This question is from an exam for a previous syllabus, and may contain

minor differences in marking or structure.

$x^3 + y^3 - 3xy = 0$

$3x^2 + 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 3x \frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$ ***M1A1***

Note: Differentiation wrt y is also acceptable.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y-3x^2}{3y^2-3x} \left(= \frac{y-x^2}{y^2-x}\right)$ ***(A1)***

Note: All following marks may be awarded if the denominator is correct, but the numerator incorrect.

$y^2 - x = 0$ ***M1***

EITHER

$x = y^2$

$y^6 + y^3 - 3y^3 = 0$ ***M1A1***

$y^6 - 2y^3 = 0$

$y^3(y^3 - 2) = 0$

$(y \neq 0) \therefore y = \sqrt[3]{2}$ ***A1***

$x = \left(\sqrt[3]{2}\ \right)^2 \left(= \sqrt[3]{4}\ \right)$ ***A1***

OR

$x^3 + xy - 3xy = 0$ ***M1***

$x(x^2 - 2y) = 0$

$x \neq 0 \Rightarrow y = \frac{x^2}{2}$ ***A1***

$y^2 = \frac{x^4}{4}$

$x = \frac{x^4}{4}$

$x(x^3 - 4) = 0$

$(x \neq 0) \therefore x = \sqrt[3]{4}$ ***A1***

$y = \frac{\left(\sqrt[3]{4}\ \right)^2}{2} = \sqrt[3]{2}$ ***A1***

[8 marks]

Let $g(x) = p^x + q$, for $x, p, q \in \mathbb{R}, p > 1$. The point $\mathrm{A}(0, a)$ lies on the graph of g.

Let $f(x) = g^{-1}(x)$. The point B lies on the graph of f and is the reflection of point A in the line $y = x$.

7a. Write down the coordinates of B. *[2 marks]*

Markscheme

$\mathrm{B}(a, 0)$ (accept $\mathrm{B}(q+1, 0)$) ***A2 N2***

[2 marks]

The line L_1 is tangent to the graph of f at B.

7b. Given that $f'(a) = \frac{1}{\ln p}$, find the equation of L_1 **in terms of** x, p and q. *[5 marks]*

Markscheme

Note: There are many approaches to this part, and the steps may be done in any order. Please check working and award marks in line with the markscheme, noting that candidates may work with the equation of the line before finding a.

FINDING a

valid attempt to find an expression for a in terms of q ***(M1)***

$g\left(0\right)=a,\ p^0+q=a$

$a=q+1$ ***(A1)***

FINDING THE EQUATION OF L_1

EITHER

attempt to substitute tangent gradient and coordinates into equation of straight line ***(M1)***

eg $y-0=f'\left(a\right)\left(x-a\right),\ y=f'\left(a\right)\left(x-\left(q+1\right)\right)$

correct equation in terms of a and p ***(A1)***

eg $y-0=\frac{1}{\ln(p)}\left(x-a\right)$

OR

attempt to substitute tangent gradient and coordinates to find b

eg $0=\frac{1}{\ln(p)}\left(a\right)+b$

$b=\frac{-a}{\ln(p)}$ ***(A1)***

THEN (must be in terms of **both** p and q)

$y=\frac{1}{\ln p}\left(x-q-1\right),\ y=\frac{1}{\ln p}x-\frac{q+1}{\ln p}$ ***A1 N3***

Note: Award ***A0*** for final answers in the form $L_1=\frac{1}{\ln p}\left(x-q-1\right)$

[5 marks]

7c. The line L_2 is tangent to the graph of g at A and has equation $y = (\ln p)\,x + q + 1$. *[7 marks]*

The line L_2 passes through the point $(-2,\ -2)$.

The gradient of the normal to g at A is $\frac{1}{\ln\left(\frac{1}{3}\right)}$.

Find the equation of L_1 in terms of x.

Markscheme

Note: There are many approaches to this part, and the steps may be done in any order. Please check working and award marks in line with the markscheme, noting that candidates may find q in terms of p before finding a value for p.

FINDING p

valid approach to find the gradient of the tangent ***(M1)***

eg $m_1 m_2 = -1, \ -\frac{1}{\frac{1}{\ln\left(1/3\right)}}, \ -\ln\left(\frac{1}{3}\right), \ -\frac{1}{\ln p} = \frac{1}{\ln\left(1/3\right)}$

correct application of log rule (seen anywhere) ***(A1)***

eg $\ln\left(\frac{1}{3}\right)^{-1}, \ -\left(\ln\left(1\right) - \ln\left(3\right)\right)$

correct equation (seen anywhere) ***A1***

eg $\ln p = \ln 3, \ p = 3$

FINDING q

correct substitution of $(-2, -2)$ into L_2 equation ***(A1)***

eg $-2 = (\ln p)(-2) + q + 1$

$q = 2\ln p - 3, \ q = 2\ln 3 - 3$ (seen anywhere) ***A1***

FINDING L_1

correct substitution of **their** p and q into **their** L_1 ***(A1)***

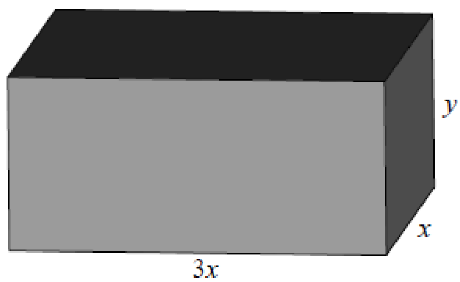
eg $y = \frac{1}{\ln 3}(x - (2\ln 3 - 3) - 1)$

$y = \frac{1}{\ln 3}(x - 2\ln 3 + 2), \ y = \frac{1}{\ln 3}x - \frac{2\ln 3 - 2}{\ln 3}$ ***A1 N2***

Note: Award ***A0*** for final answers in the form $L_1 = \frac{1}{\ln 3}(x - 2\ln 3 + 2)$.

[7 marks]

A small cuboid box has a rectangular base of length $3x$ cm and width x cm, where $x > 0$. The height is y cm, where $y > 0$.



The sum of the length, width and height is 12 cm.

8a. Write down an expression for y in terms of x. *[1 mark]*

Markscheme

$y = 12 - 4x$ ***A1 N1***

[1 mark]

The volume of the box is V cm^3.

8b. Find an expression for V in terms of x. *[2 marks]*

Markscheme

correct substitution into volume formula ***(A1)***

eg $3x \times x \times y$, $x \times 3x \times (12 - x - 3x)$, $(12 - 4x)(x)(3x)$

$V = 3x^2(12 - 4x)\ \left(= 36x^2 - 12x^3\right)$ ***A1 N2***

Note: Award ***A0*** for unfinished answers such as $3x^2(12 - x - 3x)$.

[2 marks]

8c. Find $\frac{\mathrm{d}V}{\mathrm{d}x}$. *[2 marks]*

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}x} = 72x - 36x^2$ ***A1A1 N2***

Note: Award ***A1*** for $72x$ and ***A1*** for $-36x^2$.

[2 marks]

8d. Find the value of x for which V is a maximum. *[4 marks]*

Markscheme

valid approach to find maximum ***(M1)***

eg $V' = 0,\ 72x - 36x^2 = 0$

correct working ***(A1)***

eg $x(72 - 36x),\ \frac{-72 \pm \sqrt{72^2 - 4 \cdot (-36) \cdot 0}}{2(-36)},\ 36x = 72,\ 36x(2 - x) = 0$

$x = 2$ ***A2 N2***

Note: Award ***A1*** for $x = 2$ and $x = 0$.

[4 marks]

8e. Justify your answer. *[3 marks]*

Markscheme

valid approach to explain that V is maximum when $x = 2$ ***(M1)***

eg attempt to find V'', sign chart (must be labelled V')

correct value/s ***A1***

eg $V''(2) = 72 - 72 \times 2,\ V'(a)$ where $a < 2$ **and** $V'(b)$ where $b > 2$

correct reasoning ***R1***

eg $V''(2) < 0,\ V'$ is positive for $x < 2$ **and** negative for $x > 2$

Note: Do not award ***R1*** unless ***A1*** has been awarded.

V is maximum when $x = 2$ ***AG N0***

[3 marks]

8f. Find the maximum volume. *[2 marks]*

Markscheme

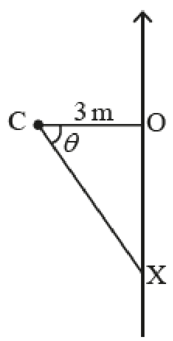
correct substitution into **their** expression for volume ***A1***

eg $3 \times 2^2 (12 - 4 \times 2),\ 36\left(2^2\right) - 12\left(2^3\right)$

$V = 48$ (cm^3) ***A1 N1***

[2 marks]

9. A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at $t = 0$. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera. *[6 marks]*



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $\text{O}\hat{\text{C}}\text{X} = \theta$.

Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let OX = x

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{d\theta}{dx} = \frac{1}{3} \times \frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{d\theta}{dt} = 24 \times \frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

METHOD 2

$\frac{dx}{dt} = 24$ (or −24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{d\theta}{dt} = \frac{dx}{dt}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{d\theta}{dt} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{8}{1+\left(\frac{d}{3}-8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

THEN

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

[6 marks]

10. Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[9 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt at implicit differentiation ***M1***

$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 = 0$ ***A1A1***

Note: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$3y^2 - 3x^2 = 0$

$\Rightarrow y = \pm x$ ***A1***

substitute either variable into original equation ***M1***

$y = x \Rightarrow x^3 = 9 \Rightarrow x = \sqrt[3]{9}$ (or $y^3 = 9 \Rightarrow y = \sqrt[3]{9}$) ***A1***

$y = -x \Rightarrow x^3 = 27 \Rightarrow x = 3$ (or $y^3 = -27 \Rightarrow y = -3$) ***A1***

$\left(\sqrt[3]{9}\ ,\ \sqrt[3]{9}\right)$, (3, −3) ***A1***

[9 marks]

The curve C is given by the equation $y = x\tan\left(\frac{\pi xy}{4}\right)$.

11a. At the point (1, 1) , show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to differentiate implicitly ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\left(\frac{\pi}{4}x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\pi}{4}y\right)\right] + \tan\left(\frac{\pi xy}{4}\right)$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x = 1$, $y = 1$ into their equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\pi}{2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\pi}{2} + 1$

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$ ***AG***

[5 marks]

11b. Hence find the equation of the normal to C at the point (1, 1). *[2 marks]*

Markscheme

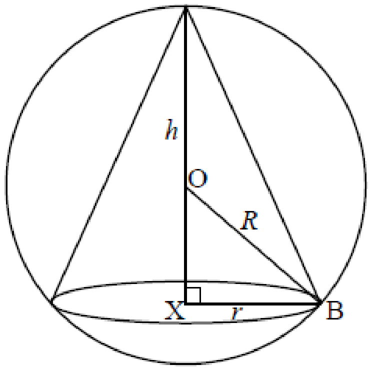
attempt to use gradient of normal $= \frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ ***(M1)***

$= \frac{\pi-2}{\pi+2}$

so equation of normal is $y - 1 = \frac{\pi-2}{\pi+2}(x - 1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1***

[2 marks]

A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.



12a. Show that the volume of the cone may be expressed by $V = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to use Pythagoras in triangle OXB ***M1***

$\Rightarrow r^2 = R^2 - (h - R)^2$ ***A1***

substitution of their r^2 into formula for volume of cone $V = \frac{\pi r^2 h}{3}$ ***M1***

$= \frac{\pi h}{3}\left(R^2 - (h - R)^2\right)$

$= \frac{\pi h}{3}\left(R^2 - \left(h^2 + R^2 - 2hR\right)\right)$ ***A1***

Note: This **A** mark is independent and may be seen anywhere for the correct expansion of $(h - R)^2$.

$= \frac{\pi h}{3}\left(2hR - h^2\right)$

$= \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ ***AG***

[4 marks]

12b. Given that there is one inscribed cone having a maximum volume, show that the volume of this cone is $\frac{32\pi R^3}{81}$. *[4 marks]*

Markscheme

at max, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$ ***R1***

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{3}\left(4Rh - 3h^2\right)$

$\Rightarrow 4Rh = 3h^2$

$\Rightarrow h = \frac{4R}{3}$ (since $h \neq 0$) ***A1***

EITHER

$V_{\max} = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ from part (a)

$= \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$ ***A1***

$= \frac{\pi}{3}\left(2R\frac{16R^2}{9} - \left(\frac{64R^3}{27}\right)\right)$ ***A1***

OR

$r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2$

$r^2 = R^2 - \frac{R^2}{9} = \frac{8R^2}{9}$ ***A1***

$\Rightarrow V_{\max} = \frac{\pi r^2}{3}\left(\frac{4R}{3}\right)$

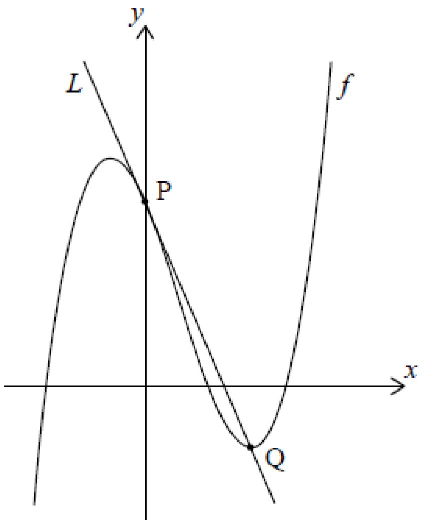
$= \frac{4\pi R}{9}\left(\frac{8R^2}{9}\right)$ ***A1***

THEN

$= \frac{32\pi R^3}{81}$ ***AG***

[4 marks]

Let $f(x) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.



The graph of f crosses the y-axis at the point P. The line *L* is tangent to the graph of f at P.

13a. Find $f'(x)$. *[2 marks]*

Markscheme

$f' = 3x^2 - 4x + a$ ***A2 N2***

[2 marks]

13b. Hence, find the equation of *L* in terms of a. *[4 marks]*

Markscheme

valid approach ***(M1)***

eg $f'(0)$

correct working ***(A1)***

eg $3(0)^2 - 4(0) + a$, slope = a, $f'(0) = a$

attempt to substitute gradient and coordinates into linear equation ***(M1)***

eg $y - 6 = a(x - 0)$, $y - 0 = a(x - 6)$, $6 = a(0) + c$, *L* $= ax + 6$

correct equation ***A1 N3***

eg $y = ax + 6$, $y - 6 = ax$, $y - 6 = a(x - 0)$

[4 marks]

13c. The graph of f has a local minimum at the point Q. The line *L* passes through Q. *[8 marks]*

Find the value of a.

Markscheme

valid approach to find intersection ***(M1)***

eg $f(x) = L$

correct equation ***(A1)***

eg $x^3 - 2x^2 + ax + 6 = ax + 6$

correct working ***(A1)***

eg $x^3 - 2x^2 = 0,\ x^2(x-2) = 0$

$x = 2$ at Q ***(A1)***

valid approach to find minimum ***(M1)***

eg $f'(x) = 0$

correct equation ***(A1)***

eg $3x^2 - 4x + a = 0$

substitution of **their** value of x at Q into **their** $f'(x) = 0$ equation ***(M1)***

eg $3(2)^2 - 4(2) + a = 0,\ 12 - 8 + a = 0$

$a = -4$ ***A1 N0***

[8 marks]

14. Use l'Hôpital's rule to find $\lim\limits_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$. *[5 marks]*

Markscheme

attempt to differentiate numerator and denominator ***M1***

$\lim\limits_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$

$= \lim\limits_{x \to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x = 0$ ***(M1)***

$= \frac{2}{3}$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

Consider the curve C defined by $y^2 = \sin(xy), y \neq 0$.

15a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt at implicit differentiation ***M1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(xy)\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right]$ ***A1M1A1***

Note: Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) + y\cos(xy)$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) = y\cos(xy)$

$\frac{\mathrm{d}y}{\mathrm{d}x}(2y - x\ \cos(xy)) = y\cos(xy)$ ***M1***

Note: Award ***M1*** for collecting derivatives and factorising.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$ ***AG***

[5 marks]

15b. Prove that, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = \pm 1$. *[5 marks]*

Markscheme

setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$y \cos(xy) = 0$ ***(M1)***

$(y \neq 0) \Rightarrow \cos(xy) = 0$ ***A1***

$\Rightarrow \sin(xy)\left(= \pm\sqrt{1 - \cos^2(xy)} = \pm\sqrt{1 - 0}\right) = \pm 1$ ***OR***

$xy = (2n+1)\frac{\pi}{2}\ (n \in \mathbb{Z})$ ***OR*** $xy = \frac{\pi}{2},\ \frac{3\pi}{2}, \ldots$ ***A1***

Note: If they offer values for xy, award ***A1*** for at least two correct values in two different 'quadrants' and no incorrect values.

$y^2 (= \sin(xy)) > 0$ ***R1***

$\Rightarrow y^2 = 1$ ***A1***

$\Rightarrow y = \pm 1$ ***AG***

[5 marks]

15c. Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[5 marks]*

Markscheme

$y = \pm 1 \Rightarrow 1 = \sin(\pm x) \Rightarrow \sin x = \pm 1$ OR
$y = \pm 1 \Rightarrow 0 = \cos(\pm x) \Rightarrow \cos x = 0$ ***(M1)***

$(\sin x = 1 \Rightarrow)\left(\frac{\pi}{2},\ 1\right),\ \left(\frac{5\pi}{2},\ 1\right)$ ***A1A1***

$(\sin x = -1 \Rightarrow)\left(\frac{3\pi}{2},\ -1\right),\ \left(\frac{7\pi}{2},\ -1\right)$ ***A1A1***

Note: Allow 'coordinates' expressed as $x = \frac{\pi}{2},\ y = 1$ for example.

Note: Each of the ***A*** marks may be awarded independently and are not dependent on ***(M1)*** being awarded.

Note: Mark only the candidate's first two attempts for each case of $\sin x$.

[5 marks]

16. Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} - 3x$ at the point where $x = 0$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$(x = 0 \Rightarrow) y = 1$ ***(A1)***

appreciate the need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \right) 2\mathrm{e}^{2x} - 3$ ***A1***

$(x = 0 \Rightarrow) \frac{\mathrm{d}y}{\mathrm{d}x} = -1$ ***A1***

$\frac{y-1}{x-0} = -1 \quad (y = 1 - x)$ ***A1***

[5 marks]

17. Use l'Hôpital's rule to determine the value of $\lim_{x \to 0}\left(\frac{2x\cos\left(x^2\right)}{5\tan x}\right)$. *[5 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to apply l'Hôpital's rule on $\lim_{x \to 0}\left(\frac{2x\cos\left(x^2\right)}{5\tan x}\right)$ **M1**

$= \lim_{x \to 0}\left(\frac{2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right)}{5\sec^2 x}\right)$ **M1A1A1**

Note: Award **M1** for attempting to use product and chain rule differentiation on the numerator, **A1** for a correct numerator and **A1** for a correct denominator. The awarding of **A1** for the denominator is independent of the **M1**.

$= \frac{2}{5}$ **A1**

[5 marks]

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