Differential Calculus revision 2 *[163 marks]*

A curve has equation $3x - 2y^2 e^{x-1} = 2$.

1a. Find an expression for $\frac{dy}{dx}$ in terms of x and y. *[5 marks]*

1b. Find the equations of the tangents to this curve at the points where the curve intersects the line $x = 1$. *[4 marks]*

Consider the curves C_1 and C_2 defined as follows

$C_1 : xy = 4$, $x > 0$

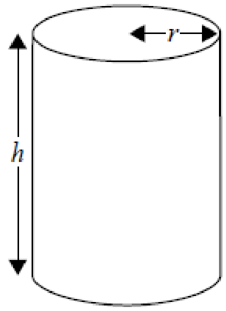
$C_2 : y^2 - x^2 = 2$, $x > 0$

2a. Using implicit differentiation, or otherwise, find $\frac{dy}{dx}$ for each curve in terms of x and y. *[4 marks]*

2b. Let P(a, b) be the unique point where the curves C_1 and C_2 intersect. Show that the tangent to C_1 at P is perpendicular to the tangent to C_2 at P. *[2 marks]*

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20π cm^3.

diagram not to scale



3a. Express *h* in terms of *r.* *[2 marks]*

The material for the base and top of the can costs 10 cents per cm^2 and the material for the curved side costs 8 cents per cm^2. The total cost of the material, in cents, is C.

3b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. *[4 marks]*

3c. Given that there is a minimum value for C, find this minimum value in terms of π. *[9 marks]*

Let $f(x) = \frac{2-3x^5}{2x^3},\ x \in \mathbb{R},\ x \neq 0$.

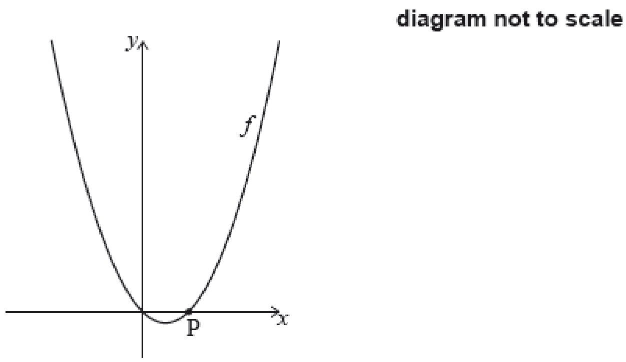
4a. The graph of $y = f(x)$ has a local maximum at A. Find the coordinates of A. *[5 marks]*

4b. Show that there is exactly one point of inflexion, B, on the graph of $y = f(x)$. *[5 marks]*

4c. The coordinates of B can be expressed in the form B$\left(2^a,\ b \times 2^{-3a}\right)$ where a, $b \in \mathbb{Q}$. Find the value of a and the value of b. *[3 marks]*

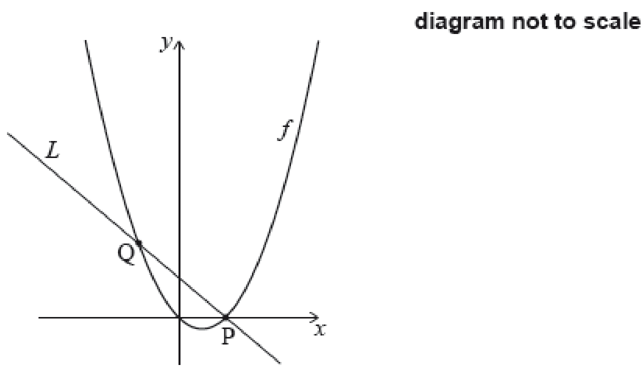
4d. Sketch the graph of $y = f(x)$ showing clearly the position of the points A and B. *[4 marks]*

Let $f(x) = x^2 - x$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.



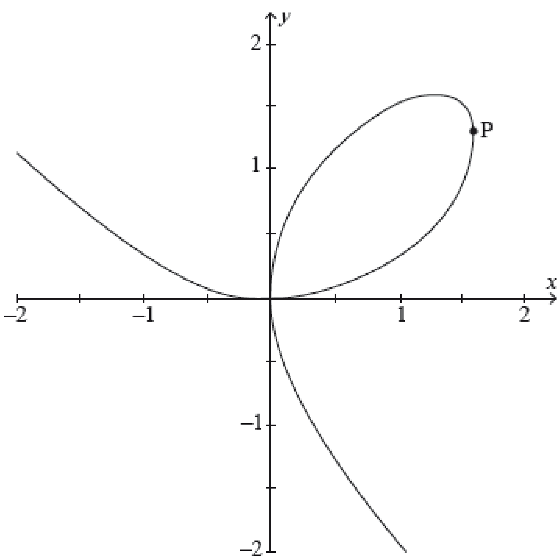
The graph of f crosses the x-axis at the origin and at the point $\mathrm{P}(1, 0)$.

The line L intersects the graph of f at another point Q, as shown in the following diagram.



5. Find the area of the region enclosed by the graph of f and the line L. *[6 marks]*

6. The folium of Descartes is a curve defined by the equation $x^3 + y^3 - 3xy = 0$, shown in the following diagram. *[8 marks]*



Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the y-axis.

Let $g(x) = p^x + q$, for $x, p, q \in \mathbb{R}, p > 1$. The point $\mathrm{A}(0, a)$ lies on the graph of g.

Let $f(x) = g^{-1}(x)$. The point B lies on the graph of f and is the reflection of point A in the line $y = x$.

7a. Write down the coordinates of B. *[2 marks]*

The line L_1 is tangent to the graph of f at B.

7b. Given that $f'(a) = \frac{1}{\ln p}$, find the equation of L_1 **in terms of** x, p and q. *[5 marks]*

7c. The line L_2 is tangent to the graph of g at A and has equation $y = (\ln p)x + q + 1$. *[7 marks]*

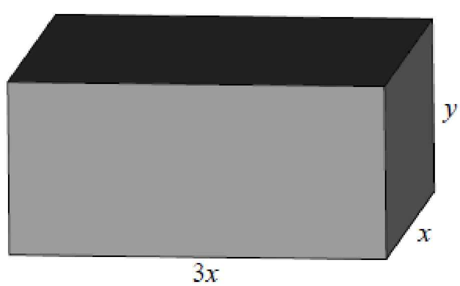
The line L_2 passes through the point $(-2, -2)$.

The gradient of the normal to g at A is $\frac{1}{\ln\left(\frac{1}{3}\right)}$.

Find the equation of L_1 in terms of x.

A small cuboid box has a rectangular base of length $3x$ cm and width x cm, where $x > 0$. The height is y cm, where $y > 0$.

diagram not to scale



The sum of the length, width and height is 12 cm.

8a. Write down an expression for y in terms of x. *[1 mark]*

The volume of the box is V cm^3.

8b. Find an expression for V in terms of x. *[2 marks]*

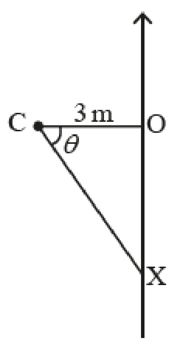
8c. Find $\frac{\mathrm{d}V}{\mathrm{d}x}$. *[2 marks]*

8d. Find the value of x for which V is a maximum. *[4 marks]*

8e. Justify your answer. *[3 marks]*

8f. Find the maximum volume. *[2 marks]*

9. A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at t = 0. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera. *[6 marks]*



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $\mathrm{O\hat{C}X} = \theta$.

Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

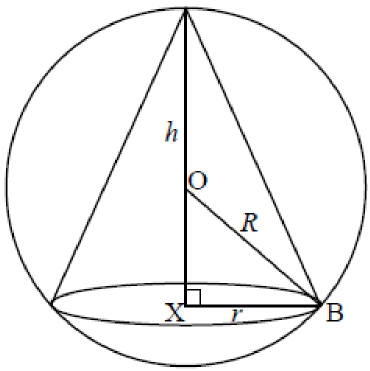
10. Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[9 marks]*

The curve C is given by the equation $y = x \tan\left(\frac{\pi xy}{4}\right)$.

11a. At the point (1, 1) , show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$. *[5 marks]*

11b. Hence find the equation of the normal to C at the point (1, 1). *[2 marks]*

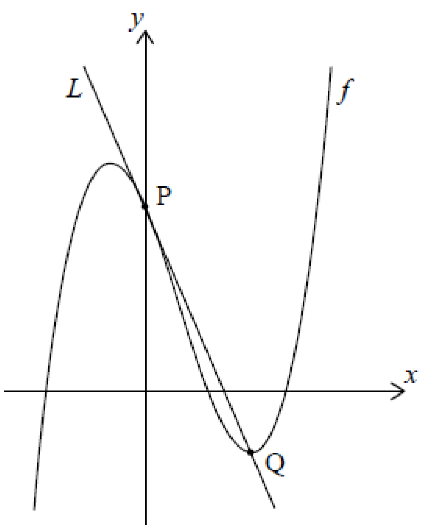
A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.



12a. Show that the volume of the cone may be expressed by $V = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$. *[4 marks]*

12b. Given that there is one inscribed cone having a maximum volume, show that the volume of this cone is $\frac{32\pi R^3}{81}$. *[4 marks]*

Let $f\left(x\right) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.



The graph of f crosses the y-axis at the point P. The line *L* is tangent to the graph of f at P.

13a. Find $f'\left(x\right)$. *[2 marks]*

13b. Hence, find the equation of *L* in terms of a. *[4 marks]*

13c. The graph of f has a local minimum at the point Q. The line *L* passes through Q. *[8 marks]*

Find the value of a.

14. Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$. *[5 marks]*

Consider the curve C defined by $y^2 = \sin(xy), y \neq 0$.

15a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$. *[5 marks]*

15b. Prove that, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = \pm 1$. *[5 marks]*

15c. Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[5 marks]*

16. Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} - 3x$ at the point where $x = 0$. *[5 marks]*

17. Use l'Hôpital's rule to determine the value of $\lim_{x \to 0}\left(\frac{2x\cos(x^2)}{5\tan x}\right)$. *[5 marks]*