Related rates *[69 marks]*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

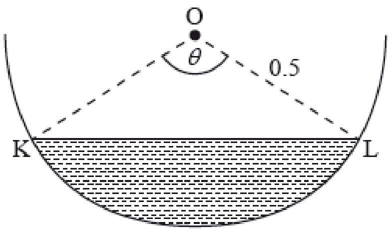


diagram not to scale

1a. Find an expression for the volume of water $V\ (\mathrm{m}^3)$ in the trough in terms of θ. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V = \text{area of segment} \times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

The volume of water is increasing at a constant rate of $0.0008\mathrm{m}^3\mathrm{s}^{-1}$.

1b. Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. *[4 marks]*

Markscheme

METHOD 1

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{5}{4}(1 - \cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.00128\ (\mathrm{rad}\, s^{-1})$ ***A1***

METHOD 2

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ ***(M1)***

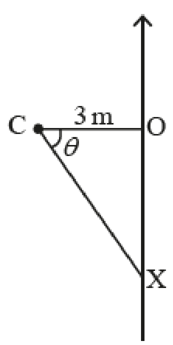
$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.00128\left(\frac{4}{3125}\right)(\mathrm{rad}\ s^{-1})$ ***A1***

[4 marks]

2. A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at $t = 0$. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera. *[6 marks]*



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let O$\hat{\mathrm{C}}$X = θ.

Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let OX = x

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{3} \times \frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 24 \times \frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d-24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(=\frac{d}{3}-8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3}-8t\right)$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{8}{1+\left(\frac{d}{3}-8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

THEN

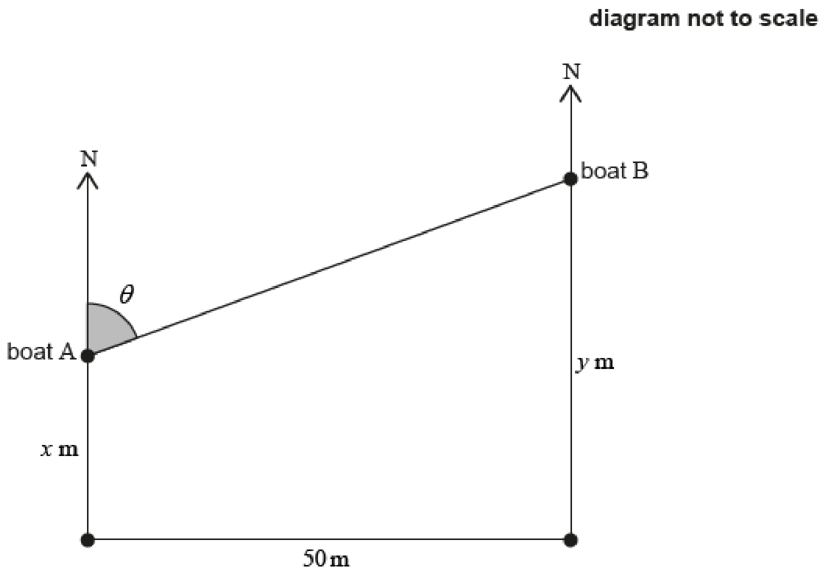
$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

[6 marks]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.



3a. Show that $y = x + 50 \cot \theta$. *[1 mark]*

Markscheme

$\tan \theta = \frac{50}{y-x}$ OR $\cot \theta = \frac{y-x}{50}$ ***A1***

$y = x + 50 \cot \theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

3b. At time T, the following conditions are true. *[6 marks]*

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

Find the speed of boat A at time T.

Markscheme

attempt to differentiate with respect to t ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots^{\circ})$ OR

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\frac{26}{25}\right) \times -0.1$

So the speed of boat A is $5.2 \left(\mathrm{ms}^{-1}\right)$ ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]

The curve C has equation $\mathrm{e}^{2y} = x^3 + y$.

4a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y} - 1}$. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts implicit differentiation on both sides of the equation **M1**

$2\mathrm{e}^{2y}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$\left(2\mathrm{e}^{2y} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ **A1**

so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y} - 1}$ **AG**

[3 marks]

4b. The tangent to C at the point P is parallel to the y-axis. *[4 marks]*

Find the x-coordinate of P.

Markscheme

attempts to solve $2e^{2y} - 1 = 0$ for y **(M1)**

$y = -0.346\ldots \left(= \frac{1}{2}\ln\frac{1}{2}\right)$ **A1**

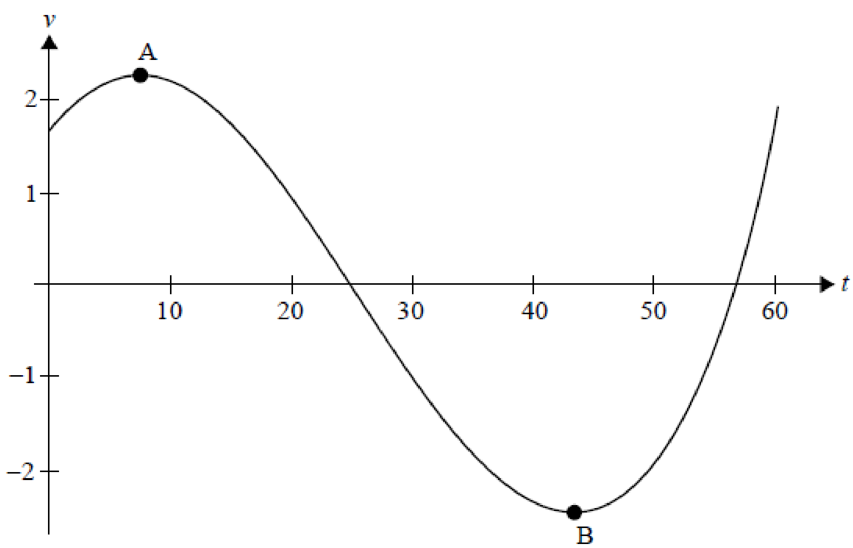
attempts to solve $e^{2y} = x^3 + y$ for x given their value of y **(M1)**

$x = 0.946\left(= \left(\frac{1}{2}\left(1 - \ln\frac{1}{2}\right)\right)^{\frac{1}{3}}\right)$ **A1**

[4 marks]

A body moves in a straight line such that its velocity, $v\,\mathrm{ms}^{-1}$, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \leqslant t \leqslant 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



5a. Determine the coordinates of point A and the coordinates of point B. *[4 marks]*

Markscheme

$\mathrm{A}\,(7.47,\,2.28)$ and $\mathrm{B}\,(43.4,\,-2.45)$ ***A1A1A1A1***

[4 marks]

5b. Hence, write down the maximum speed of the body. *[1 mark]*

Markscheme

maximum speed is $2.45\ \left(\mathrm{ms}^{-1}\right)$ ***A1***

[1 mark]

The body first comes to rest at time $t = t_1$. Find

5c. the value of t_1. *[2 marks]*

Markscheme

$v = 0 \Rightarrow t_1 = 25.1\ (\mathrm{s})$ ***(M1)A1***

[2 marks]

5d. the distance travelled between $t = 0$ and $t = t_1$. *[2 marks]*

Markscheme

$\int_0^{t_1} v\,\mathrm{d}t$ ***(M1)***

$= 41.0\ (\mathrm{m})$ ***A1***

[2 marks]

5e. the acceleration when $t = t_1$. *[2 marks]*

Markscheme

$a = \frac{\mathrm{d}v}{\mathrm{d}t}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200\ \left(\mathrm{ms}^{-2}\right)$ ***A1***

Note: Accept $a = -0.2$.

[2 marks]

5f. Find the distance travelled in the first 30 seconds. *[3 marks]*

Markscheme

attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v| \, \mathrm{d}t$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v \, \mathrm{d}t$ ***(A1)***

THEN

$= 43.3 \ (\mathrm{m})$ ***A1***

[3 marks]

A point P moves in a straight line with velocity v ms^{-1} given by $v\left(t\right) = \mathrm{e}^{-t} - 8t^2\mathrm{e}^{-2t}$ at time t seconds, where $t \geq 0$.

6a. Determine the first time t_1 at which P has zero velocity. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to solve $v\left(t\right) = 0$ for t or equivalent ***(M1)***

t_1 = 0.441(s) ***A1***

[2 marks]

6b. Find an expression for the acceleration of P at time t. *[2 marks]*

Markscheme

$a(t) = \frac{dv}{dt} = -e^{-t} - 16te^{-2t} + 16t^2e^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

[2 marks]

6c. Find the value of the acceleration of P at time t_1. *[1 mark]*

Markscheme

$a(t_1) = -2.28$ (ms^{-2}) ***A1***

[1 mark]

Xavier, the parachutist, jumps out of a plane at a height of h metres above the ground. After free falling for 10 seconds his parachute opens. His velocity, $v\,\mathrm{ms}^{-1}$, t seconds after jumping from the plane, can be modelled by the function

$$v(t) = \begin{cases} 9.8t, & 0 \leqslant t \leqslant 10 \\ \frac{98}{\sqrt{1+(t-10)^2}}, & t > 10 \end{cases}$$

7a. Find his velocity when $t = 15$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$v(15) = \frac{98}{\sqrt{1+(15-10)^2}}$ ***(M1)***

$v(15) = 19.2\ (\mathrm{ms}^{-1})$ ***A1***

[2 marks]

7b. Calculate the vertical distance Xavier travelled in the first 10 seconds. *[2 marks]*

Markscheme

$\int_0^{10} 9.8t \, dt$ ***(M1)***

$= 490 \text{ (m)}$ ***A1***

[2 marks]

His velocity when he reaches the ground is 2.8ms^{-1}.

7c. Determine the value of h. *[5 marks]*

Markscheme

$\frac{98}{\sqrt{1+(t-10)^2}} = 2.8$ ***(M1)***

$t = 44.985\ldots \text{ (s)}$ ***A1***

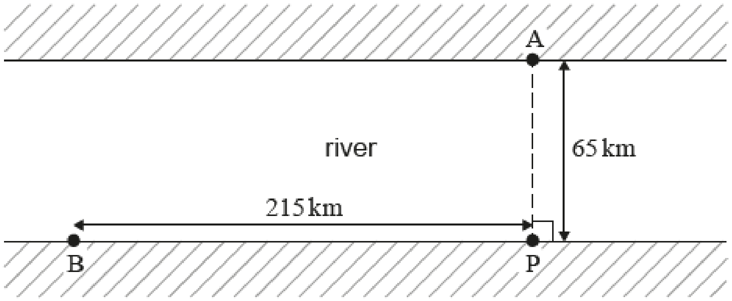
$h = 490 + \int_{10}^{44.9\ldots} \frac{98}{\sqrt{1+(t-10)^2}} dt$ ***(M1)(A1)***

$h = 906 \text{ (m)}$ ***A1***

[5 marks]

Points A and P lie on opposite banks of a river, such that AP is the shortest distance across the river. Point B represents the centre of a city which is located on the riverbank. $\mathrm{PB} = 215\ \mathrm{km}$, $\mathrm{AP} = 65\ \mathrm{km}$ and $\mathrm{A\hat{P}B} = 90^\circ$.

The following diagram shows this information.



A boat travels at an average speed of $42\ \mathrm{km\,h^{-1}}$. A bus travels along the straight road between P and B at an average speed of $84\ \mathrm{km\,h^{-1}}$.

Find the travel time, in hours, from A to B given that

8a. the boat is taken from A to P, and the bus from P to B. *[2 marks]*

Markscheme

$\frac{\mathrm{AP}}{42}$ OR $\frac{215}{84}$ OR $\frac{65}{42} + \frac{215}{84}$ ***(M1)***

time $= 4.10714\ldots$ (hours)

time $= 4.11$ (hours) ***A1***

[2 marks]

8b. the boat travels directly to B. *[2 marks]*

Markscheme

$\mathrm{AB} = \sqrt{215^2 + 65^2}(= 224.610\ldots)$ ***(A1)***

time $= 5.34787\ldots$ (hours)

time $= 5.35$ (hours) ***A1***

[2 marks]

There is a point D, which lies on the road from P to B, such that $\mathrm{BD} = x\ \mathrm{km}$. The boat travels from A to D, and the bus travels from D to B.

8c. Find an expression, in terms of x for the travel time T, from A to B, passing through D. *[3 marks]*

Markscheme

$\mathrm{AD} = \sqrt{(215 - x)^2 + 65^2}$ ***(A1)***

$t = \frac{\sqrt{(215-x)^2+65^2}}{42}$ ***(A1)***

$T = \frac{\sqrt{(215-x)^2+65^2}}{42} + \frac{x}{84} \left(= \frac{\sqrt{x^2-430x+50450}}{42} + \frac{x}{84} \right)$ ***A1***

[3 marks]

8d. Find the value of x so that T is a minimum. *[2 marks]*

Markscheme

valid approach to find the minimum for T (may be seen in (iii)) ***(M1)***

graph of T OR $T\text{'} = 0$ OR graph of $T\text{'}$

$x = 177.472\ldots\ \mathrm{km}$

$x = 177\ \mathrm{km}$ ***A1***

[2 marks]

8e. Write down the minimum value of T. *[1 mark]*

Markscheme

$T = 3.89980\ldots$

$T = 3.90$ (hours) ***A1***

Note: Only allow ***FT*** in (b)(ii) and (iii) for $0 < x < 215$ and a function T that has a minimum in that interval.

[1 mark]

An excursion involves renting the boat and the bus. The cost to rent the boat is $\$ 200$ per hour, and the cost to rent the bus is $\$ 150$ per hour.

8f. Find the new value of x so that the total cost C to travel from A to B via D is a minimum. *[3 marks]*

Markscheme

$C = 200 \cdot \frac{\sqrt{(215-x)^2+65^2}}{42} + 150 \cdot \frac{x}{84}$ ***(A1)***

valid approach to find the minimum for $C(x)$ (may be seen in (ii)) ***(M1)***

graph of C OR $C' = 0$ OR graph of C'

$x = 188.706\ldots \text{ km}$

$x = 189 \text{ km}$ ***A1***

Note: Only allow ***FT*** from (b) if the function T has a minimum in $0 < x < 215$.

[3 marks]

8g. Write down the minimum total cost for this journey. *[1 mark]*

Markscheme

$C = 670.864$

$C = \$671$ ***A1***

Note: Only allow ***FT*** from (c)(i) if the function C has a minimum in $0 < x < 215$.

[1 mark]