Related rates *[69 marks]*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

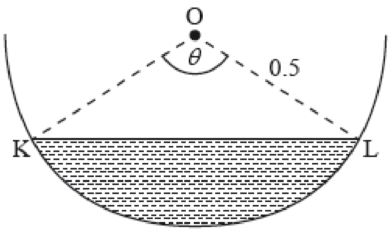


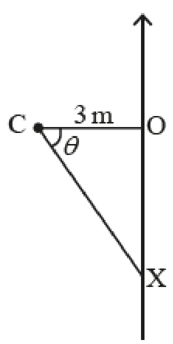
diagram not to scale

1a. Find an expression for the volume of water $V\ (\mathrm{m}^3)$ in the trough in terms of θ. *[3 marks]*

The volume of water is increasing at a constant rate of $0.0008\mathrm{m}^3\mathrm{s}^{-1}$.

1b. Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. *[4 marks]*

2. A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at t = 0. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera. *[6 marks]*



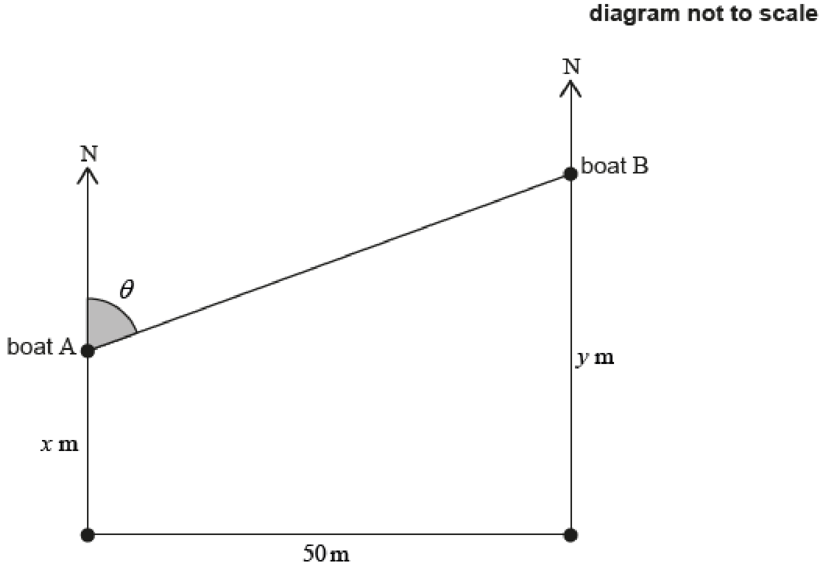
A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $\mathrm{O\hat{C}X} = \theta$.

Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.



3a. Show that $y = x + 50 \cot \theta$. *[1 mark]*

3b. At time T, the following conditions are true. *[6 marks]*

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

Find the speed of boat A at time T.

The curve C has equation $\mathrm{e}^{2y} = x^3 + y$.

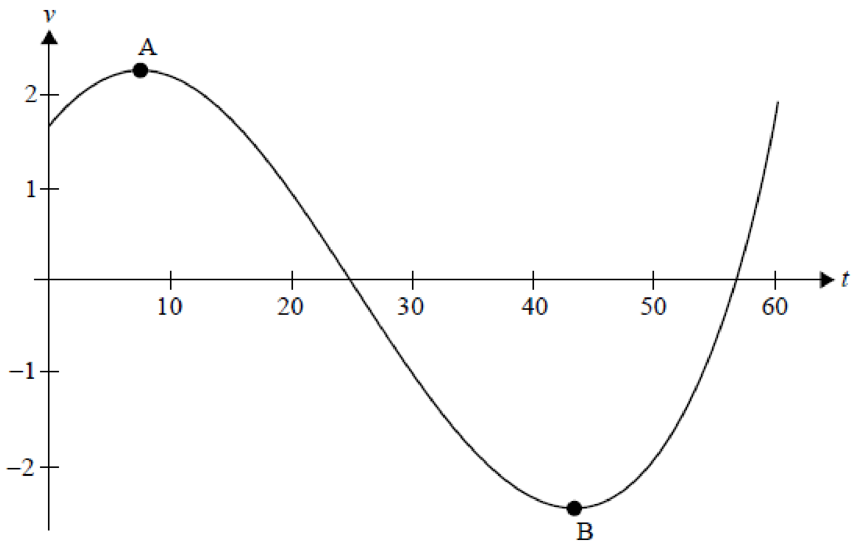
4a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y}-1}$. *[3 marks]*

4b. The tangent to C at the point P is parallel to the y-axis. *[4 marks]*

Find the x-coordinate of P.

A body moves in a straight line such that its velocity, $v\,\mathrm{ms}^{-1}$, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \leqslant t \leqslant 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



5a. Determine the coordinates of point A and the coordinates of point B. *[4 marks]*

5b. Hence, write down the maximum speed of the body. *[1 mark]*

The body first comes to rest at time $t = t_1$. Find

5c. the value of t_1. *[2 marks]*

5d. the distance travelled between $t = 0$ and $t = t_1$. *[2 marks]*

5e. the acceleration when $t = t_1$. *[2 marks]*

5f. Find the distance travelled in the first 30 seconds. *[3 marks]*

A point P moves in a straight line with velocity v ms^{-1} given by $v(t) = \mathrm{e}^{-t} - 8t^2\mathrm{e}^{-2t}$ at time t seconds, where $t \geq 0$.

6a. Determine the first time t_1 at which P has zero velocity. *[2 marks]*

6b. Find an expression for the acceleration of P at time *t*. *[2 marks]*

6c. Find the value of the acceleration of P at time t_1. *[1 mark]*

Xavier, the parachutist, jumps out of a plane at a height of h metres above the ground. After free falling for 10 seconds his parachute opens. His velocity, $v\,\text{ms}^{-1}$, t seconds after jumping from the plane, can be modelled by the function

$$v(t) = \begin{cases} 9.8t, & 0 \leqslant t \leqslant 10 \\ \frac{98}{\sqrt{1+(t-10)^2}}, & t > 10 \end{cases}$$

7a. Find his velocity when $t = 15$. *[2 marks]*

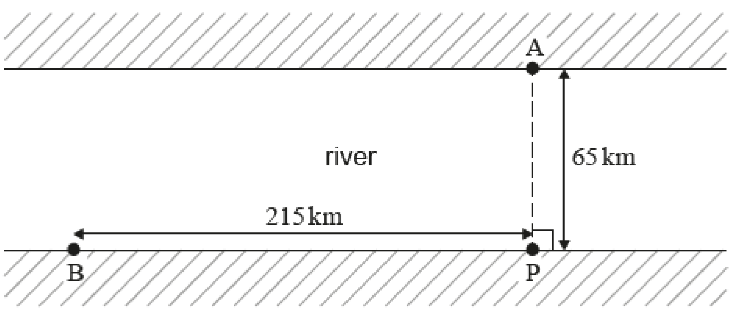
7b. Calculate the vertical distance Xavier travelled in the first 10 seconds. *[2 marks]*

His velocity when he reaches the ground is 2.8ms^{-1}.

7c. Determine the value of h. *[5 marks]*

Points A and P lie on opposite banks of a river, such that AP is the shortest distance across the river. Point B represents the centre of a city which is located on the riverbank. $\text{PB} = 215\text{ km}$, $\text{AP} = 65\text{ km}$ and $\text{A}\widehat{\text{P}}\text{B} = 90^{\circ}$.

The following diagram shows this information.



A boat travels at an average speed of 42 kmh^{-1}. A bus travels along the straight road between P and B at an average speed of 84 kmh^{-1}.

Find the travel time, in hours, from A to B given that

8a. the boat is taken from A to P, and the bus from P to B. *[2 marks]*

8b. the boat travels directly to B. *[2 marks]*

There is a point D, which lies on the road from P to B, such that $\mathrm{BD} = x\ \mathrm{km}$. The boat travels from A to D, and the bus travels from D to B.

8c. Find an expression, in terms of x for the travel time T, from A to B, passing through D. *[3 marks]*

8d. Find the value of x so that T is a minimum. *[2 marks]*

8e. Write down the minimum value of T. *[1 mark]*

An excursion involves renting the boat and the bus. The cost to rent the boat is $\$\,200$ per hour, and the cost to rent the bus is $\$\,150$ per hour.

8f. Find the new value of x so that the total cost C to travel from A to B via D is a minimum. *[3 marks]*

8g. Write down the minimum total cost for this journey. *[1 mark]*