1. (a) Show that lines $\frac{x-2}{1}=\frac{y-2}{3}=\frac{z-3}{1}$ and $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection.

(8)

(b) Find the Cartesian equation of the plane Π that contains the two lines.

(6)

(c) The point Q(3, 4, 3) lies on Π. The line L passes through the midpoint of [PQ]. Point S is on L such that $|\overrightarrow{PS}| = |\overrightarrow{QS}| = 3$, and the triangle PQS is normal to the plane Π. Given that there are two possible positions for S, find their coordinates.

(15)

(Total 29 marks)

(a) $L_1: x = 2 + \lambda; y = 2 + 3\lambda; z = 3 + \lambda$ (A1)

$L_2: x = 2 + \mu; y = 3 + 4\mu; z = 4 + 2\mu$ (A1)

At the point of intersection (M1)

$2 + \lambda = 2 + \mu$ (1)

$2 + 3\lambda = 3 + 4\mu$ (2)

$3 + \lambda = 4 + 2\mu$ (3)

From (1), $\lambda = \mu$ A1

Substituting in (2), $2 + 3\lambda = 3 + 4\lambda$

$\Rightarrow \lambda = \mu = -1$ A1

We need to show that these values satisfy (3). (M1)

They do because LHS = RHS = 2; therefore the lines intersect. R1

So P is (1, –1, 2). A1 N3

(b) The normal to Π is normal to both lines. It is therefore given by the vector product of the two direction vectors.

Therefore, normal vector is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{vmatrix}$ M1A1

$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ A2

The Cartesian equation of Π is $2x - y + z = 2 + 1 + 2$ (M1)

i.e. $2x - y + z = 5$ A1 N2

(c) The midpoint M of [PQ] is $\left(2, \frac{3}{2}, \frac{5}{2}\right)$. M1A1

The direction of $\overline{\text{MS}}$ is the same as the normal to Π, i.e. $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ (R1)

The coordinates of a general point R on $\overline{\text{MS}}$ are therefore

$\left(2 + 2\lambda, \frac{3}{2} - \lambda, \frac{5}{2} + \lambda\right)$ (M1)

It follows that $\overline{\text{PR}} = (1 + 2\lambda)\boldsymbol{i} + \left(\frac{5}{2} - \lambda\right)\boldsymbol{j} + \left(\frac{1}{2} + \lambda\right)\boldsymbol{k}$ A1A1A1

At S, length of $\overline{\text{PR}}$ is 3, i.e. (M1)

$(1 + 2\lambda)^2 + \left(\frac{5}{2} - \lambda\right)^2 + \left(\frac{1}{2} + \lambda\right)^2 = 9$ A1

$1 + 4\lambda + 4\lambda^2 + \frac{25}{4} - 5\lambda + \lambda^2 + \frac{1}{4} + \lambda + \lambda^2 = 9$ (A1)

$6\lambda^2 = \frac{6}{4}$ A1

$\lambda = \pm\frac{1}{2}$ A1

Substituting these values, (M1)

the possible positions of S are (3, 1, 3) and (1, 2, 2) A1A1 N2

[29]

2. Consider the points A(1, 2, 1), B(0, –1, 2), C(1, 0, 2) and D(2, –1, –6).

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(2)

(b) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$.

(2)

(c) Hence, or otherwise find the area of triangle ABC.

(3)

(d) Find the Cartesian equation of the plane *P* containing the points A, B and C.

(3)

(e) Find a set of parametric equations for the line *L* through the point D and perpendicular to the plane *P*.

(3)

(f) Find the point of intersection E, of the line *L* and the plane *P*.

(4)

(g) Find the distance from the point D to the plane *P*.

(2)

(h) Find a unit vector that is perpendicular to the plane *P*.

(2)

(i) The point F is a reflection of D in the plane *P*. Find the coordinates of F.

(4)

(Total 25 marks)

(a) $\overrightarrow{AB} = -\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{BC} = \boldsymbol{i} + \boldsymbol{j}$ A1A1

(b) $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ M1

$= -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ A1

(c) Area of $\Delta ABC = \frac{1}{2}\left|-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\right|$ M1A1

$= \frac{1}{2}\sqrt{1+1+4}$

$= \frac{\sqrt{6}}{2}$ A1

(d) A normal to the plane is given by $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{BC} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ (M1)

Therefore, the equation of the plane is of the form $-x + y + 2z = g$
and since the plane contains A, then $-1 + 2 + 2 = g \Rightarrow g = 3$. M1

Hence, an equation of the plane is $-x + y + 2z = 3$. A1

(e) Vector $\boldsymbol{n}$ above is parallel to the required line.

Therefore, $x = 2 - t$ A1

$y = -1 + t$ A1

$z = -6 + 2t$ A1

(f) $x = 2 - t$

$y = -1 + t$

$z = -6 + 2t$

$-x + y + 2z = 3$

$-2 + t - 1 + t - 12 + 4t = 3$ M1A1

$-15 + 6t = 3$

$6t = 18$

$t = 3$ A1

Point of intersection $(-1, 2, 0)$ A1

(g) Distance $= \sqrt{3^2 + 3^2 + 6^2} = \sqrt{54}$ (M1)A1

(h) Unit vector in the direction of $\boldsymbol{n}$ is $\mathrm{e} = \frac{1}{|\boldsymbol{n}|} \times \boldsymbol{n}$ (M1)

$= \frac{1}{\sqrt{6}}(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ A1

Note: $-\mathrm{e}$ is also acceptable.

(i) Point of intersection of L and P is $(-1, 2, 0)$.

$\overrightarrow{DE} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ (M1)A1

$\Rightarrow \overrightarrow{EF} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ M1

$\Rightarrow$ coordinates of F are $(-4, 5, 6)$ A1

[25]

3. The acceleration in m s^{-2} of a particle moving in a straight line at time t seconds, $t \geq 0$, is given by the formula $a = -\frac{1}{2}v$. When $t = 0$, the velocity is 40 m s^{-1}.

Find an expression for v in terms of t.

(Total 6 marks)

$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{2}v$	A1
$\int \frac{\mathrm{d}v}{v} = \int -\frac{1}{2}\mathrm{d}t$	(A1)
$\ln v = -\frac{1}{2}t + c$	(A1)
$v = \mathrm{e}^{-\frac{1}{2}t+c} \quad \left(= A\mathrm{e}^{-\frac{1}{2}t}\right)$	(A1)
$t = 0,\ v = 40$, so $A = 40$	M1
$v = 40\mathrm{e}^{-\frac{1}{2}t}$ (or equivalent)	A1

[6]

4. Calculate $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

(Total 6 marks)

Let $f(x) = \frac{\sin x - x}{x \sin x}$	(M1)
$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right)$	A1A1
$= \lim_{x \to 0}\left(\frac{-\sin x}{2\cos x - x\sin x}\right)$	A1A1
$= 0$	A1 N2

[6]

5. (a) Show that the solution of the differential equation

$$\frac{dy}{dx} = \cos x \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = \pi$, is $y = \arctan(1 + \sin x)$.

(5)

(b) Determine the value of the constant a for which the following limit exists

$$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - a}{\left(x - \frac{\pi}{2}\right)^2}$$

and evaluate that limit.

(12)
(Total 17 marks)

(a) this separable equation has general solution

$\int \sec^2 y \, dy = \int \cos x \, dx$ (M1)(A1)

$\tan y = \sin x + c$ A1

the condition gives

$\tan \frac{\pi}{4} = \sin \pi + c \Rightarrow c = 1$ M1

the solution is $\tan y = 1 + \sin x$ A1

$y = \arctan(1 + \sin x)$ AG

(b) the limit cannot exist unless $a = \arctan\left(1 + \sin \frac{\pi}{2}\right) = \arctan 2$ R1A1

in that case the limit can be evaluated using l'Hopital's rule (twice)

limit is

$\lim_{x \to \frac{\pi}{2}} \frac{(\arctan(1 + \sin x))'}{2\left(x - \frac{\pi}{2}\right)} = \lim_{x \to \frac{\pi}{2}} \frac{y'}{2\left(x - \frac{\pi}{2}\right)}$ M1A1

where y is the solution of the differential equation

the numerator has zero limit (from the factor $\cos x$ in the differential equation) R1

so required limit is

$\lim_{x \to \frac{\pi}{2}} = \frac{y''}{2}$ M1A1

finally,

$y'' = -\sin x \cos^2 y - 2 \cos x \cos y \sin y \times y'(x)$ M1A1

since $\cos y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}$ A1

$y'' = -\frac{1}{5}$ at $x = \frac{\pi}{2}$ A1

the required limit is $-\frac{1}{10}$ A1

[17]

6. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}y}+\frac{xy}{4-x^2}=1$, where $|x|<2$ and $y=1$ when $x=0$.

(a) Use Euler's method with $h=0.25$, to find an approximate value of y when $x=1$, giving your answer to two decimal places.

(10)

(b) (i) By first finding an integrating factor, solve this differential equation. Give your answer in the form $y=f(x)$.

(ii) Calculate, correct to two decimal places, the value of y when $x=1$.

(10)

(c) Sketch the graph of $y=f(x)$ for $0\le x\le 1$. Use your sketch to explain why your approximate value of y is greater than the true value of y.

(4)

(Total 24 marks)

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{xy}{4-x^2}$

x	y	$\mathrm{d}y/\mathrm{d}x$	$h\times \mathrm{d}y/\mathrm{d}x$	
0	1	1	0.25	A2
0.25	1.25	0.9206349206	0.2301587302	A2
0.5	1.48015873	0.8026455027	0.2006613757	A2
0.75	1.680820106	0.6332756132	0.1583189033	A2
1	1.839139009			A1

To two decimal places, when $x=1$, $y=1.84$. A1 N0

(b) (i) Integrating factor = $e^{\int\left(\frac{x}{4-x^2}\right)dx}$ (M1)

$= e^{\left(-\frac{1}{2}\ln(4-x^2)\right)}$ A1

$= \frac{1}{\sqrt{4-x^2}}$ A1

It follows that $\frac{d}{dx}\left(\frac{y}{\sqrt{4-x^2}}\right) = \frac{1}{\sqrt{4-x^2}}$ (M1)

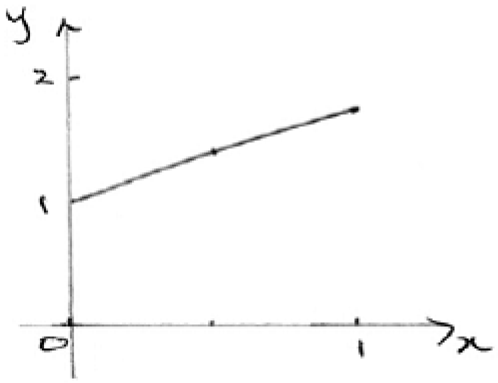
$\frac{y}{\sqrt{4-x^2}} = \arcsin\left(\frac{x}{2}\right) + C$ A1A1

Putting $x = 0, y = 1, \Rightarrow \frac{1}{2} = C$ A1

Therefore, $y = \sqrt{4-x^2}\left(\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\right)$ A2 N0

(ii) When $x = 1, y = 1.77.$ A1 N1

(c)



A2

Since $\frac{dy}{dx}$ is decreasing the value of y is over-estimated at each step. R1A1

[24]