1. Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

$(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2 = \sin^2\theta - (1 - \cos\theta)^2 + \mathrm{i}\,2\sin\theta(1 - \cos\theta)$ M1A1

Let α be the required argument.

$\tan\alpha = \dfrac{2\sin\theta(1-\cos\theta)}{\sin^2\theta - (1-\cos\theta)^2}$ M1

$= \dfrac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta)-(1-2\cos\theta+\cos^2\theta)}$ (M1)

$= \dfrac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ A1

$= \tan\theta$ A1

$\alpha = \theta$ A1

[7]

2. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\dfrac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)

(Total 12 marks)

(a) $z = (1 - \mathrm{i})^{\frac{1}{4}}$

Let $1 - \mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\Rightarrow r = \sqrt{2}$ A1

$\theta = -\frac{\pi}{4}$ A1

$$z = \left(\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right)^{\frac{1}{4}}$$ M1

$$= \left(\sqrt{2}\left(\cos\left(-\frac{\pi}{4} + 2n\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{4} + 2n\pi\right)\right)\right)^{\frac{1}{4}}$$

$$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{\pi}{16} + \frac{n\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{16} + \frac{n\pi}{2}\right)\right)$$ M1

$$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{\pi}{16}\right)\right)$$

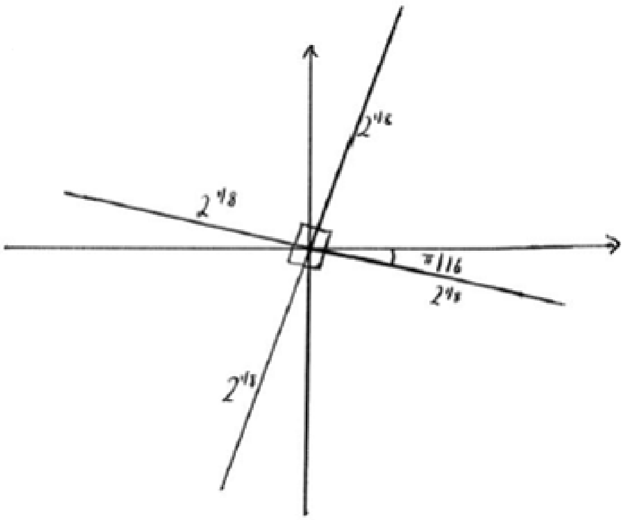
Note: Award M1 above for this line if the candidate has forgotten to add 2π and no other solution given.

$$= 2^{\frac{1}{8}}\left(\cos\left(\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)$$

$$= 2^{\frac{1}{8}}\left(\cos\left(\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)$$

$$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{9\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{16}\right)\right)$$ A2

(b)



A2

Note: Award A1 for roots being shown equidistant from the origin and one in each quadrant.
A1 for correct angular positions. It is not necessary to see written evidence of angle, but must agree with the diagram.

(c) $\frac{z_2}{z_1} = \frac{2^{\frac{1}{8}}\left(\left(\cos\frac{15\pi}{16}\right)+\mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)}{2^{\frac{1}{8}}\left(\left(\cos\frac{7\pi}{16}\right)+\mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)}$ M1A1

$= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$ (A1)

$= \mathrm{i}$ A1 N2

$(\Rightarrow a = 0, b = 1)$

[12]

3. Consider the complex geometric series $\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta} + \frac{1}{4}\mathrm{e}^{3\mathrm{i}\theta} + \ldots$

(a) Find an expression for z, the common ratio of this series. **(2)**

(b) Show that $|z| < 1$. **(2)**

(c) Write down an expression for the sum to infinity of this series. **(2)**

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

(10)
(Total 16 marks)

(a) $z = \dfrac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}}$ (M1)

$z = \frac{1}{2}e^{i\theta}$ A1 N2

(b) $|z| = \frac{1}{2}$ A2

$|z| < 1$ AG

(c) Using $S_\infty = \dfrac{a}{1-r}$ (M1)

$S_\infty = \dfrac{e^{i\theta}}{1 - \frac{1}{2}e^{i\theta}}$ A1 N2

(d) (i) $S_\infty = \dfrac{e^{i\theta}}{1 - \frac{1}{2}e^{i\theta}} = \dfrac{\text{cis}\,\theta}{1 - \frac{1}{2}\text{cis}\,\theta}$ (M1)

$\dfrac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$ (A1)

Also $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

$= \text{cis}\,\theta + \frac{1}{2}\text{cis}2\theta + \frac{1}{4}\text{cis}3\theta + \ldots$ (M1)

$S_\infty = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$ A1

(ii) Taking real parts,

$$\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos 3\theta+\ldots=\text{Re}\left(\frac{\cos\theta+\text{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\text{i}\sin\theta)}\right)$$ A1

$$=\text{Re}\left(\frac{(\cos\theta+\text{i}\sin\theta)}{\left(1-\frac{1}{2}\cos\theta-\frac{1}{2}\text{i}\sin\theta\right)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\text{i}\sin\theta}{\left(1-\frac{1}{2}\cos\theta+\frac{1}{2}\text{i}\sin\theta\right)}\right)$$ M1

$$=\frac{\cos\theta-\frac{1}{2}\cos^2\theta-\frac{1}{2}\sin^2\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\frac{1}{4}\sin^2\theta}$$ A1

$$=\frac{\left(\cos\theta-\frac{1}{2}\right)}{1-\cos\theta+\frac{1}{4}(\sin^2\theta+\cos^2\theta)}$$ A1

$$=\frac{(2\cos\theta-1)\div 2}{(4-4\cos\theta+1)\div 4}=\frac{4(2\cos\theta-1)}{2(5-4\cos\theta)}$$ A1

$$=\frac{4\cos\theta-2}{5-4\cos\theta}$$ A1AG N0

[25]

4. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $re^{\text{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \text{i}\sin n\theta = (\cos\theta + \text{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + \text{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)
(Total 31 marks)

5. Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(5)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(7)
(Total 16 marks)

(a) $z = \dfrac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3}$ M1

$-1 + i\sqrt{3} = re^{i\theta} \Rightarrow r = 2$ A1

$\theta = \arctan \dfrac{\sqrt{3}}{-1} = \dfrac{2\pi}{3}$ A1

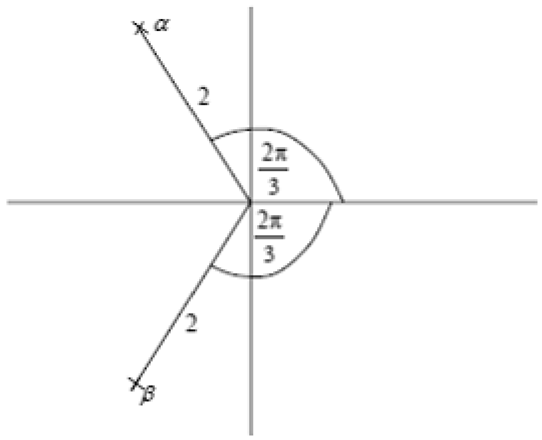
$-1 - i\sqrt{3} = re^{i\theta} \Rightarrow r = 2$

$\theta = \arctan \dfrac{\sqrt{3}}{-1} = -\dfrac{2\pi}{3}$ A1

$\Rightarrow \alpha = 2e^{i\frac{2\pi}{3}}$ A1

$\Rightarrow \beta = 2e^{-i\frac{2\pi}{3}}$ A1

(b)



A1A1

(c) $\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$

Let $n = 1$

Left hand side = $\cos 1\theta + i \sin 1\theta = \cos\theta + i \sin\theta$

Right hand side = $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

Hence true for $n = 1$ M1A1

Assume true for $n = k$ M1

$\cos k\theta + i \sin k\theta = (\cos\theta + i\sin\theta)^k$

$\Rightarrow \cos(k+1)\theta + i\sin(k+1)\theta = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$ M1A1

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta \cos\theta - \sin k\theta \sin\theta + i(\cos k\theta \sin\theta + \sin k\theta \cos\theta)$ A1

$= \cos(k+1)\theta + i\sin(k+1)\theta$ A1

Hence if true for $n = k$, true for $n = k + 1$

However if it is true for $n = 1$

$\Rightarrow$ true for $n = 2$ *etc.* R1

$\Rightarrow$ hence proved by induction

(d) $\frac{\alpha^3}{\beta^2} = \frac{8e^{i2\pi}}{4e^{-i\frac{4\pi}{3}}} = 2e^{i\frac{4\pi}{3}}$ A1

$= 2\cos\frac{4\pi}{3} + 2i\sin\frac{4\pi}{3}$ (M1)

$= -\frac{2}{2} - 2\frac{i\sqrt{3}}{2} = -1 - i\sqrt{3}$ A1A1

(e) $\alpha^3 = 8e^{i2\pi}$ A1

$\beta^3 = 8e^{-i2\pi}$ A1

Since $e^{2\pi}$ and $e^{-2\pi}$ are the same $\alpha^3 = \beta^3$ R1

(f) **EITHER**

$\alpha = -1 + i\sqrt{3}$ $\beta = -1 - i\sqrt{3}$

$\alpha^* = -1 - i\sqrt{3}$ $\beta^* = -1 + i\sqrt{3}$ A1

$\alpha\beta^* = (-1 + i\sqrt{3})(-1 + i\sqrt{3}) = 1 - 2i\sqrt{3} - 3 = 2 - 2i\sqrt{3}$ M1A1

$\beta\alpha^* = (-1 - i\sqrt{3})(-1 - i\sqrt{3}) = 1 + 2i\sqrt{3} - 3 = -2 + 2i\sqrt{3}$ A1

$\Rightarrow \alpha\beta^* + \beta\alpha^* = -4$ A1

OR

Since $\alpha^* = \beta$ and $\beta^* = \alpha$

$\alpha\beta^* = 2e^{i\frac{2\pi}{3}} \times 2e^{i\frac{2\pi}{3}} = 4e^{i\frac{4\pi}{3}}$ M1A1

$\beta\alpha^* = 2e^{-i\frac{2\pi}{3}} \times 2e^{-i\frac{2\pi}{3}} = 4e^{-i\frac{4\pi}{3}}$ A1

$\alpha\beta^* + \beta\alpha^* = 4\left(e^{i\frac{4\pi}{3}} + e^{-i\frac{4\pi}{3}}\right)$

$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$ A1

$= 8\cos\frac{4\pi}{3} = 8 \times -\frac{1}{2} = -4$ A1

(g) $\alpha^n = 2^n e^{i2\frac{\pi n}{3}}$ M1A1

This is real when n is a multiple of 3 R1

i.e. $n = 3N$ where $N \in \mathbb{Z}^+$

[31]

6. The complex number z is defined as $z = \cos\theta + i\sin\theta$.

(a) State de Moivre's theorem.

(1)

(b) Show that $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$.

(3)

(c) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form.

(3)

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(4)

(e) Check that your result in part (d) is true for $\theta = \frac{\pi}{4}$.

(4)

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \,\mathrm{d}\theta$.

(4)

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5 \theta \,\mathrm{d}\theta$, explaining your reasoning.

(3)
(Total 22 marks)

(a) any appropriate form, e.g. $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ A1

(b) $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ A1

$\frac{1}{z^n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ (M1)

$= \cos n\theta - \mathrm{i}\sin(n\theta)$ A1

therefore $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$ AG

(c) $\left(z - \frac{1}{z}\right)^5 = z^5 + \binom{5}{1}z^4\left(-\frac{1}{z}\right) + \binom{5}{2}z^3\left(-\frac{1}{z}\right)^2 + \binom{5}{3}z^2\left(-\frac{1}{z}\right)^3 + \binom{5}{4}z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$

(M1)(A1)

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ A1

(d) $\left(z - \frac{1}{z}\right)^5 = z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ M1A1

$(2\mathrm{i}\sin\theta)^5 = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$ M1A1

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ AG

(e) $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\text{LHS} = 16\left(\sin\frac{\pi}{4}\right)^5$

$= 16\left(\frac{\sqrt{2}}{2}\right)^5$

$= 2\sqrt{2}\ \left(= \frac{4}{\sqrt{2}}\right)$ A1

$\text{RHS} = \sin\left(\frac{5\pi}{4}\right) - 5\sin\left(\frac{3\pi}{4}\right) + 10\sin\left(\frac{\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} - 5\left(\frac{\sqrt{2}}{2}\right) + 10\left(\frac{\sqrt{2}}{2}\right)$ M1A1

Note: Award M1 for attempted substitution.

$= 2\sqrt{2}\ \left(= \frac{4}{\sqrt{2}}\right)$ A1

hence this is true for $\theta = \frac{\pi}{4}$ AG

(f) $\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta) d\theta$ M1

$= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{\pi}{2}}$ A1

$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$ A1

$= \frac{8}{15}$ A1

(g) $\int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta = \frac{8}{15}$, with appropriate reference to symmetry and graphs. A1R1R1

Note: Award first R1 for partially correct reasoning e.g. sketches of graphs of sin and cos.
Award second R1 for fully correct reasoning involving $\sin^5$ and $\cos^5$.

[22]

7. Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$.

(3)

(b) Show that $(w - 1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(3)

(c) **Hence** show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(6)
(Total 12 marks)

(a) **EITHER**

$w^5 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$ (M1)

$= \cos 2\pi + i \sin 2\pi$ A1

$= 1$ A1

Hence w is a root of $z^5 - 1 = 0$ AG

OR

Solving $z^5 = 1$ (M1)

$z = \cos\frac{2\pi}{5}n + i\sin\frac{2\pi}{5}n, \quad n = 0, 1, 2, 3, 4.$ A1

$n = 1$ gives $\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ which is w A1

(b) $(w - 1)(1 + w + w^2 + w^3 + w^4) = w + w^2 + w^3 + w^4 + w^5 - 1 - w - w^2 - w^3 - w^4$ M1

$= w^5 - 1$ A1

Since $w^5 - 1 = 0$ **and** $w \neq 1$, $w^4 + w^3 + w^2 + w + 1 = 0$. R1

(c) $1 + w + w^2 + w^3 + w^4 =$

$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 +$

$\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3 + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^4$ (M1)

$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} +$

$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} + \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$ M1

$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} +$

$\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$ M1A1A1

Notes: Award M1 for attempting to replace 6π and 8π by 4π and 2π
Award A1 for correct cosine terms and A1 for correct sine terms.

$= 1 + 2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5} = 0$ A1

Note: Correct methods involving equating real parts, use of conjugates or reciprocals are also accepted.

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ AG

Note: Use of cis notation is acceptable throughout this question.

[12]