1. Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

2. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)

(Total 12 marks)

3. Consider the complex geometric series $\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta} + \frac{1}{4}\mathrm{e}^{3\mathrm{i}\theta} + \ldots$.

(a) Find an expression for z, the common ratio of this series.

(2)

(b) Show that $|z| < 1$.

(2)

(c) Write down an expression for the sum to infinity of this series.

(2)

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

(10)

(Total 16 marks)

4. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + \mathrm{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)
(Total 31 marks)

5. Consider $\omega = \cos\left(\dfrac{2\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(5)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \dfrac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(7)
(Total 16 marks)

6. The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) State de Moivre's theorem.

(1)

(b) Show that $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$.

(3)

(c) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^5$ giving your answer in simplified form.

(3)

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(4)

(e) Check that your result in part (d) is true for $\theta = \dfrac{\pi}{4}$.

(4)

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \mathrm{d}\theta$.

(4)

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5\theta \, \mathrm{d}\theta$, explaining your reasoning.

(3)
(Total 22 marks)

7. Let $w = \cos\dfrac{2\pi}{5} + \mathrm{i}\sin\dfrac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$.

(3)

(b) Show that $(w - 1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(3)

(c) **Hence** show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$.

(6)
(Total 12 marks)