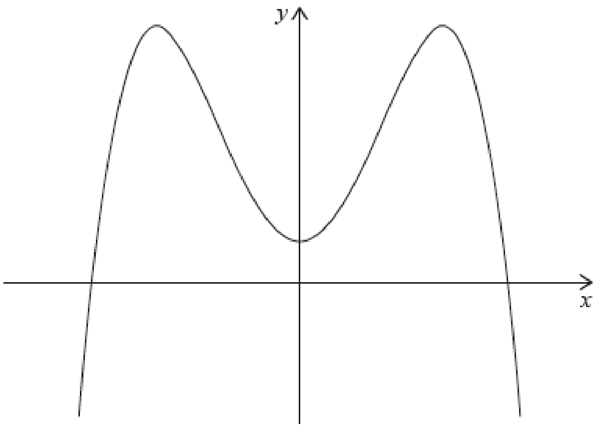
Calculus SL *[81 marks]*

Consider the function $f(x) = -x^4 + ax^2 + 5$, where a is a constant. Part of the graph of $y = f(x)$ is shown below.



1a. Write down the y-intercept of the graph. *[1 mark]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

5 ***(A1)***

Note: Accept an answer of $(0,\ 5)$.

[1 mark]

1b. Find $f'(x)$. *[2 marks]*

Markscheme

$(f'(x) =) - 4x^3 + 2ax$ ***(A1)(A1)***

Note: Award ***(A1)*** for $-4x^3$ and ***(A1)*** for $+2ax$. Award at most ***(A1)(A0)*** if extra terms are seen.

[2 marks]

It is known that at the point where $x = 2$ the tangent to the graph of $y = f(x)$ is horizontal.

1c. Show that $a = 8$. *[2 marks]*

Markscheme

$-4 \times 2^3 + 2a \times 2 = 0$ ***(M1)(M1)***

Note: Award ***(M1)*** for substitution of $x = 2$ into their derivative, ***(M1)*** for equating their derivative, written in terms of a, to 0 leading to a correct answer (note, the 8 does not need to be seen).

$a = 8$ ***(AG)***

[2 marks]

1d. Find $f(2)$. *[2 marks]*

Markscheme

$(f(2) =) - 2^4 + 8 \times 2^2 + 5$ ***(M1)***

Note: Award ***(M1)*** for correct substitution of $x = 2$ and $a = 8$ into the formula of the function.

21 ***(A1)(G2)***

[2 marks]

There are two other points on the graph of $y = f(x)$ at which the tangent is horizontal.

1e. Write down the x-coordinates of these two points; *[2 marks]*

Markscheme

$(x =)\ -2,\ (x =)\ 0$ ***(A1)(A1)***

Note: Award ***(A1)*** for each correct solution. Award at most ***(A0)(A1)*****(ft)** if answers are given as $(-2\ , 21)$ and $(0,\ 5)$ or $(-2,\ 0)$ and $(0,\ 0)$.

[2 marks]

1f. Write down the intervals where the gradient of the graph of $y = f(x)$ is positive. *[2 marks]*

Markscheme

$x < -2,\ 0 < x < 2$ ***(A1)*(ft)*(A1)*(ft)**

Note: Award ***(A1)*(ft)** for $x < -2$, follow through from part (d)(i) provided their value is negative.

Award ***(A1)*(ft)** for $0 < x < 2$, follow through only from their 0 from part (d)(i); 2 must be the upper limit.

Accept interval notation.

[2 marks]

1g. Write down the range of $f(x)$. *[2 marks]*

Markscheme

$y \leqslant 21$ ***(A1)*(ft)*(A1)***

Notes: Award ***(A1)*(ft)** for 21 seen in an interval or an inequality, ***(A1)*** for "$y \leqslant$".

Accept interval notation.

Accept $-\infty < y \leqslant 21$ or $f(x) \leqslant 21$.

Follow through from their answer to part (c)(ii). Award at most ***(A1)*(ft)*(A0)*** if x is seen instead of y. Do not award the second ***(A1)*** if a (finite) lower limit is seen.

[2 marks]

1h. Write down the number of possible solutions to the equation $f(x) = 5$. *[1 mark]*

Markscheme

3 (solutions) ***(A1)***

[1 mark]

1i. The equation $f(x) = m$, where $m \in \mathbb{R}$, has four solutions. Find the possible values of m. *[2 marks]*

Markscheme

$5 < m < 21$ or equivalent ***(A1)*****(ft)*****(A1)***

Note: Award ***(A1)*****(ft)** for 5 and 21 seen in an interval or an inequality, ***(A1)*** for correct strict inequalities. Follow through from their answers to parts (a) and (c)(ii).

Accept interval notation.

[2 marks]

Let $f(x) = x\mathrm{e}^{-x}$ and $g(x) = -3f(x) + 1$.

The graphs of f and g intersect at $x = p$ and $x = q$, where $p < q$.

2a. Find the value of p and of q. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid attempt to find the intersection ***(M1)***

$egf = g$, sketch, one correct answer

$p = 0.357402,\ q = 2.15329$

$p = 0.357,\ q = 2.15$ ***A1A1*** ***N3***

[3 marks]

2b. Hence, find the area of the region enclosed by the graphs of f and g. *[3 marks]*

Markscheme

attempt to set up an integral involving subtraction (in any order) ***(M1)***

eg $\int_p^q [f(x) - g(x)]\,\mathrm{d}x,\ \int_p^q f(x)\mathrm{d}x - \int_p^q g(x)\mathrm{d}x$

0.537667

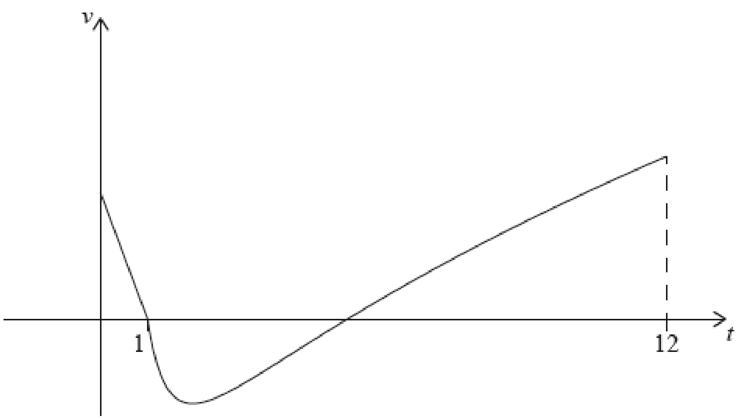
$\text{area} = 0.538$ ***A2*** ***N3***

[3 marks]

A particle P starts from a point A and moves along a horizontal straight line. Its velocity $v \text{cm s}^{-1}$ after t seconds is given by

$$v(t) = \begin{cases} -2t+2, & \text{for} 0 \leqslant t \leqslant 1 \\ 3\sqrt{t} + \frac{4}{t^2} - 7, & \text{for} 1 \leqslant t \leqslant 12 \end{cases}$$

The following diagram shows the graph of v.



3a. Find the initial velocity of P. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid attempt to substitute $t = 0$ into the correct function ***(M1)***

eg $-2(0) + 2$

2 ***A1*** ***N2***

[2 marks]

P is at rest when $t = 1$ and $t = p$.

3b. Find the value of p. *[2 marks]*

Markscheme

recognizing $v = 0$ when P is at rest ***(M1)***

5.21834

$p = 5.22\ (\text{seconds})$ ***A1*** ***N2***

[2 marks]

When $t = q$, the acceleration of P is zero.

3c. (i) Find the value of q. *[4 marks]*

(ii) Hence, find the **speed** of P when $t = q$.

Markscheme

(i) recognizing that $a = v'$ ***(M1)***

eg $v' = 0$, minimum on graph

1.95343

$q = 1.95$ ***A1*** ***N2***

(ii) valid approach to find **their** minimum ***(M1)***

eg $v(q),\ -1.75879$, reference to min on graph

1.75879

speed $= 1.76\ (\text{c m s}^{-1})$ ***A1*** ***N2***

[4 marks]

3d. (i) Find the total distance travelled by P between $t = 1$ and $t = p$. *[6 marks]*

(ii) Hence or otherwise, find the displacement of P from A when $t = p$.

Markscheme

(i) substitution of **correct** $v(t)$ into distance formula, ***(A1)***

eg $\int_1^p \left|3\sqrt{t}+\frac{4}{t^2}-7\right| \mathrm{d}t,\ \left|\int 3\sqrt{t}+\frac{4}{t^2}-7\mathrm{d}t\right|$

4.45368

distance $= 4.45\ (\mathrm{cm})$ ***A1*** ***N2***

(ii) displacement from $t=1$ to $t=p$ (seen anywhere) ***(A1)***

eg $-4.45368,\ \int_1^p\left(3\sqrt{t}+\frac{4}{t^2}-7\right)\mathrm{d}t$

displacement from $t=0$ to $t=1$ ***(A1)***

eg $\int_0^1(-2t+2)\mathrm{d}t,\ 0.5\times 1\times 2,\ 1$

valid approach to find displacement for $0\leqslant t\leqslant p$ ***M1***

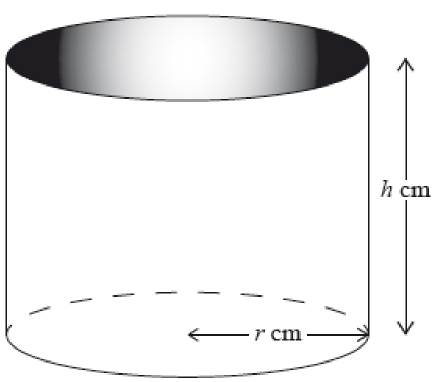
eg $\int_0^1(-2t+2)\mathrm{d}t+\int_1^p\left(3\sqrt{t}+\frac{4}{t^2}-7\right)\mathrm{d}t,\ \int_0^1(-2t+2)\mathrm{d}t-4.45$

-3.45368

displacement $= -3.45\ (\mathrm{cm})$ ***A1*** ***N2***

[6 marks]

A water container is made in the shape of a cylinder with internal height h cm and internal base radius r cm.



The water container has no top. The inner surfaces of the container are to be coated with a water-resistant material.

4a. Write down a formula for A, the surface area to be coated. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$(A =)\ \pi r^2 + 2\pi rh$ ***(A1)(A1)***

Note: Award ***(A1)*** for either πr^2 **OR** $2\pi rh$ seen. Award ***(A1)*** for two correct terms added together.

[2 marks]

The volume of the water container is $0.5\mathrm{m}^3$.

4b. Express this volume in cm^3. *[1 mark]*

Markscheme

$500\,000$ ***(A1)***

Notes: Units **not** required.

[1 mark]

4c. Write down, in terms of r and h, an equation for the volume of this water container. *[1 mark]*

Markscheme

$500\,000 = \pi r^2 h$ ***(A1)*****(ft)**

Notes: Award ***(A1)*****(ft)** for $\pi r^2 h$ equating to their part (b).

Do not accept unless $V = \pi r^2 h$ is explicitly defined as their part (b).

[1 mark]

4d. Show that $A = \pi r^2 + \frac{1\,000\,000}{r}$. *[2 marks]*

Markscheme

$A = \pi r^2 + 2\pi r \left(\frac{500\,000}{\pi r^2}\right)$ ***(A1)*****(ft)*****(M1)***

Note: Award ***(A1)*****(ft)** for their $\frac{500\,000}{\pi r^2}$ seen.

Award ***(M1)*** for correctly substituting **only** $\frac{500\,000}{\pi r^2}$ into a **correct** part (a).

Award ***(A1)*****(ft)*****(M1)*** for rearranging part (c) to $\pi r h = \frac{500\,000}{r}$ and substituting for $\pi r h$ in expression for A.

$A = \pi r^2 + \frac{1\,000\,000}{r}$ ***(AG)***

Notes: The conclusion, $A = \pi r^2 + \frac{1\,000\,000}{r}$, must be consistent with their working seen for the ***(A1)*** to be awarded.

Accept 10^6 as equivalent to $1\,000\,000$.

[2 marks]

The water container is designed so that the area to be coated is minimized.

4e. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. *[3 marks]*

Markscheme

$2\pi r - \frac{1\,000\,000}{r^2}$ ***(A1)(A1)(A1)***

Note: Award ***(A1)*** for $2\pi r$, ***(A1)*** for $\frac{1}{r^2}$ or r^{-2}, ***(A1)*** for $-1\,000\,000$.

[3 marks]

4f. Using your answer to part (e), find the value of r which minimizes A. *[3 marks]*

Markscheme

$2\pi r - \frac{1\,000\,000}{r^2} = 0$ ***(M1)***

Note: Award ***(M1)*** for equating their part (e) to zero.

$r^3 = \frac{1\,000\,000}{2\pi}$ **OR** $r = \sqrt[3]{\frac{1\,000\,000}{2\pi}}$ ***(M1)***

Note: Award ***(M1)*** for isolating r.

OR

sketch of derivative function ***(M1)***

with its zero indicated ***(M1)***

$(r =)\ 54.2\ (\text{cm})\ (54.1926\ldots)$ ***(A1)*****(ft)*****(G2)***

[3 marks]

4g. Find the value of this minimum area. *[2 marks]*

Markscheme

$\pi(54.1926\ldots)^2 + \frac{1\,000\,000}{(54.1926\ldots)}$ ***(M1)***

Note: Award ***(M1)*** for correct substitution of their part (f) into the given equation.

$= 27\,700\ (\text{cm}^2)\ (27\,679.0\ldots)$ ***(A1)*****(ft)*****(G2)***

[2 marks]

One can of water-resistant material coats a surface area of 2000cm^2.

4h. Find the least number of cans of water-resistant material that will coat the area in part (g). *[3 marks]*

Markscheme

$\frac{27\,679.0\ldots}{2000}$ ***(M1)***

Note: Award ***(M1)*** for dividing their part (g) by 2000.

$= 13.8395\ldots$ ***(A1)*****(ft)**

Notes: Follow through from part (g).

14 (cans) ***(A1)*****(ft)*****(G3)***

Notes: Final ***(A1)*** awarded for rounding up their $13.8395\ldots$ to the next integer.

[3 marks]

Let $f(x) = \ln x$ and $g(x) = 3 + \ln\left(\frac{x}{2}\right)$, for $x > 0$.

The graph of g can be obtained from the graph of f by two transformations:

$$\text{a horizontal stretch of scale factor} q \text{followed by}$$
$$\text{a translation of} \begin{pmatrix} h \\ k \end{pmatrix}.$$

5a. Write down the value of q; *[1 mark]*

Markscheme

$q = 2$ ***A1 N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

5b. Write down the value of h; *[1 mark]*

Markscheme

$h = 0$ ***A1 N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

5c. Write down the value of k. *[1 mark]*

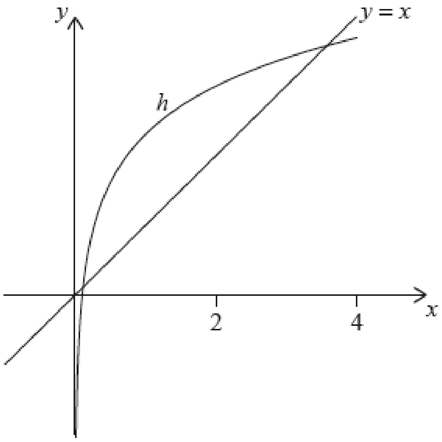
Markscheme

$k = 3$ ***A1 N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

Let $h(x) = g(x) \times \cos(0.1x)$, for $0 < x < 4$. The following diagram shows the graph of h and the line $y = x$.



The graph of h intersects the graph of h^{-1} at two points. These points have x coordinates 0.111 and 3.31 correct to three significant figures.

5d. Find $\int_{0.111}^{3.31} (h(x) - x)\,\mathrm{d}x$. *[2 marks]*

Markscheme

2.72409

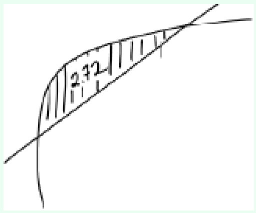
2.72 ***A2 N2***

[2 marks]

5e. Hence, find the area of the region enclosed by the graphs of h and h^{-1}. *[3 marks]*

Markscheme

recognizing area between $y = x$ and h equals 2.72 ***(M1)***

eg 

recognizing graphs of h and h^{-1} are reflections of each other in $y = x$ ***(M1)***

eg area between $y = x$ and h equals between $y = x$ and h^{-1}

$2 \times 2.72 \int_{0.111}^{3.31} \left(x - h^{-1}(x)\right) \mathrm{d}x = 2.72$

5.44819

5.45 ***A1 N3***

[??? marks]

5f. Let d be the vertical distance from a point on the graph of h to the line $y = x$. There is a point $\mathrm{P}(a, b)$ on the graph of h where d is a maximum. *[7 marks]*

Find the coordinates of P, where $0.111 < a < 3.31$.

Markscheme

valid attempt to find d ***(M1)***

eg difference in y-coordinates, $d = h(x) - x$

correct expression for d ***(A1)***

eg $\left(\ln \frac{1}{2}x + 3\right)(\cos 0.1x) - x$

valid approach to find when d is a maximum ***(M1)***

eg max on sketch of d, attempt to solve $d' = 0$

0.973679

$x = 0.974$ ***A2 N4***

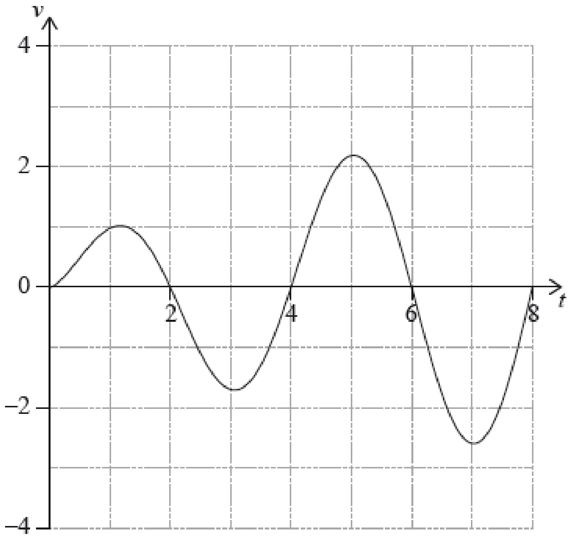
substituting **their** x value into $h(x)$ ***(M1)***

2.26938

$y = 2.27$ ***A1 N2***

[7 marks]

A particle P moves along a straight line. Its velocity $v_{\mathrm{P}}\,\mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{P}} = \sqrt{t}\sin\left(\frac{\pi}{2}t\right)$, for $0 \leqslant t \leqslant 8$. The following diagram shows the graph of v_{P}.



6a. Write down the first value of t at which P changes direction. *[1 mark]*

Markscheme

$t = 2$ ***A1 N1***

[1 mark]

6b. Find the **total** distance travelled by P, for $0 \leqslant t \leqslant 8$. *[2 marks]*

Markscheme

substitution of limits or function into formula or correct sum ***(A1)***

eg $\int_0^8 |v|\,\mathrm{d}t,\ \int |v_Q|\,\mathrm{d}t,\ \int_0^2 v\mathrm{d}t - \int_2^4 v\mathrm{d}t + \int_4^6 v\mathrm{d}t - \int_6^8 v\mathrm{d}t$

9.64782

distance $= 9.65(\mathrm{metres})$ ***A1 N2***

[2 marks]

6c. A second particle Q also moves along a straight line. Its velocity, $v_{\mathrm{Q}}\,\mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{Q}} = \sqrt{t}$ for $0 \leqslant t \leqslant 8$. After k seconds Q has travelled the same total distance as P. *[4 marks]*

Find k.

Markscheme

correct approach ***(A1)***

$egs = \int \sqrt{t},\ \int_0^k \sqrt{t}\mathrm{d}t,\ \int_0^k |v_{\mathrm{Q}}|\,\mathrm{d}t$

correct integration ***(A1)***

$eg \int \sqrt{t} = \frac{2}{3}t^{\frac{3}{2}} + c,\ \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^k,\ \frac{2}{3}k^{\frac{3}{2}}$

equating their expression to the distance travelled by their P ***(M1)***

$eg \frac{2}{3}k^{\frac{3}{2}} = 9.65,\ \int_0^k \sqrt{t}\mathrm{d}t = 9.65$

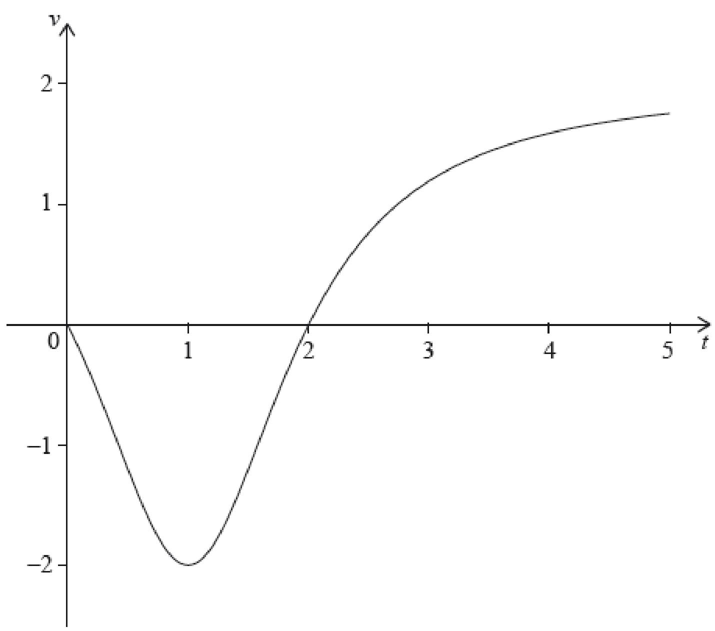
5.93855

5.94 (seconds) ***A1 N3***

[4 marks]

7. **Note: In this question, distance is in metres and time is in seconds.** *[6 marks]*

A particle moves along a horizontal line starting at a fixed point A. The velocity v of the particle, at time t, is given by $v(t) = \frac{2t^2-4t}{t^2-2t+2}$, for $0 \leqslant t \leqslant 5$. The following diagram shows the graph of v



There are t-intercepts at $(0,\ 0)$ and $(2,\ 0)$.

Find the maximum distance of the particle from A during the time $0 \leqslant t \leqslant 5$ and justify your answer.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1 (displacement)

recognizing $s = \int v\mathrm{d}t$ ***(M1)***

consideration of displacement at $t = 2$ **and** $t = 5$ (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_0^5 v$

Note: Must have both for any further marks.

correct displacement at $t = 2$ and $t = 5$ (seen anywhere) ***A1A1***

-2.28318 (accept 2.28318), 1.55513

valid reasoning comparing correct displacements ***R1***

eg $|-2.28| > |1.56|$, more left than right

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1***.

METHOD 2 (distance travelled)

recognizing distance $= \int |v|\,\mathrm{d}t$ ***(M1)***

consideration of distance travelled from $t = 0$ to 2 **and** $t = 2$ to 5 (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_2^5 v$

Note: Must have both for any further marks

correct distances travelled (seen anywhere) ***A1A1***

2.28318, (accept -2.28318), 3.83832

valid reasoning comparing correct distance values ***R1***

eg $3.84 - 2.28 < 2.28,\ 3.84 < 2 \times 2.28$

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1***.

[6 marks]