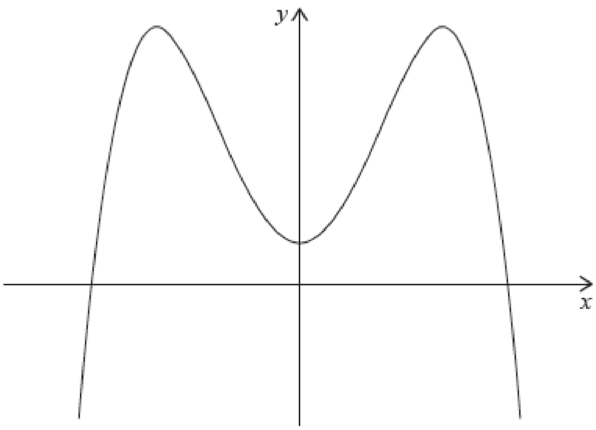
Calculus SL *[81 marks]*

Consider the function $f(x) = -x^4 + ax^2 + 5$, where a is a constant. Part of the graph of $y = f(x)$ is shown below.



1a. Write down the y-intercept of the graph. *[1 mark]*

1b. Find $f'(x)$. *[2 marks]*

It is known that at the point where $x = 2$ the tangent to the graph of $y = f(x)$ is horizontal.

1c. Show that $a = 8$. *[2 marks]*

1d. Find $f(2)$. *[2 marks]*

There are two other points on the graph of $y = f(x)$ at which the tangent is horizontal.

1e. Write down the x-coordinates of these two points; *[2 marks]*

1f. Write down the intervals where the gradient of the graph of $y = f(x)$ is positive. *[2 marks]*

1g. Write down the range of $f(x)$. *[2 marks]*

1h. Write down the number of possible solutions to the equation $f(x) = 5$. *[1 mark]*

1i. The equation $f(x) = m$, where $m \in \mathbb{R}$, has four solutions. Find the possible values of m. *[2 marks]*

Let $f(x) = x\mathrm{e}^{-x}$ and $g(x) = -3f(x) + 1$.

The graphs of f and g intersect at $x = p$ and $x = q$, where $p < q$.

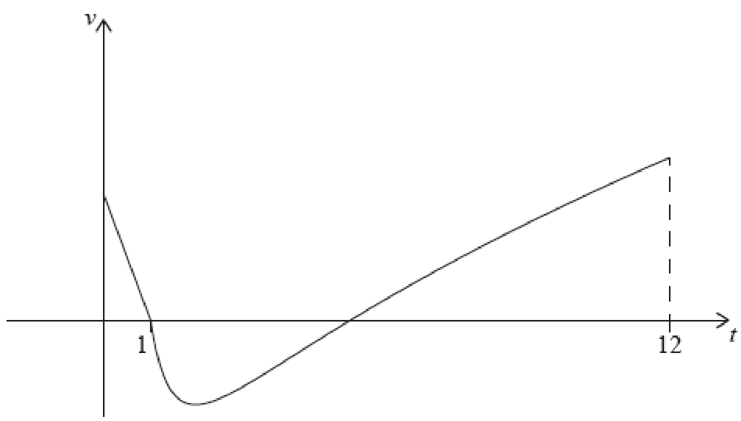
2a. Find the value of p and of q. *[3 marks]*

2b. Hence, find the area of the region enclosed by the graphs of f and g. *[3 marks]*

A particle P starts from a point A and moves along a horizontal straight line. Its velocity $v\,\mathrm{cm\,s^{-1}}$ after t seconds is given by

$$v(t) = \begin{cases} -2t + 2, & \text{for} 0 \leqslant t \leqslant 1 \\ 3\sqrt{t} + \frac{4}{t^2} - 7, & \text{for} 1 \leqslant t \leqslant 12 \end{cases}$$

The following diagram shows the graph of v.



3a. Find the initial velocity of P. *[2 marks]*

P is at rest when $t = 1$ and $t = p$.

3b. Find the value of p. *[2 marks]*

When $t = q$, the acceleration of P is zero.

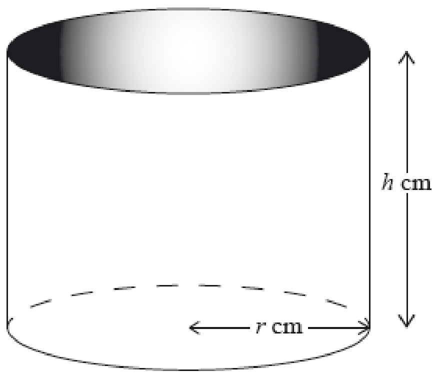
3c. (i) Find the value of q. *[4 marks]*

(ii) Hence, find the **speed** of P when $t = q$.

3d. (i) Find the total distance travelled by P between $t = 1$ and $t = p$. *[6 marks]*

(ii) Hence or otherwise, find the displacement of P from A when $t = p$.

A water container is made in the shape of a cylinder with internal height h cm and internal base radius r cm.



The water container has no top. The inner surfaces of the container are to be coated with a water-resistant material.

4a. Write down a formula for A, the surface area to be coated. *[2 marks]*

The volume of the water container is 0.5m^3.

4b. Express this volume in cm^3. *[1 mark]*

4c. Write down, in terms of r and h, an equation for the volume of this water container. *[1 mark]*

4d. Show that $A = \pi r^2 + \frac{1\,000\,000}{r}$. *[2 marks]*

The water container is designed so that the area to be coated is minimized.

4e. Find $\frac{\mathrm{d}A}{\mathrm{d}r}$. *[3 marks]*

4f. Using your answer to part (e), find the value of r which minimizes A. *[3 marks]*

4g. Find the value of this minimum area. *[2 marks]*

One can of water-resistant material coats a surface area of 2000cm^2.

4h. Find the least number of cans of water-resistant material that will coat the area in part (g). *[3 marks]*

Let $f(x) = \ln x$ and $g(x) = 3 + \ln\left(\frac{x}{2}\right)$, for $x > 0$.

The graph of g can be obtained from the graph of f by two transformations:

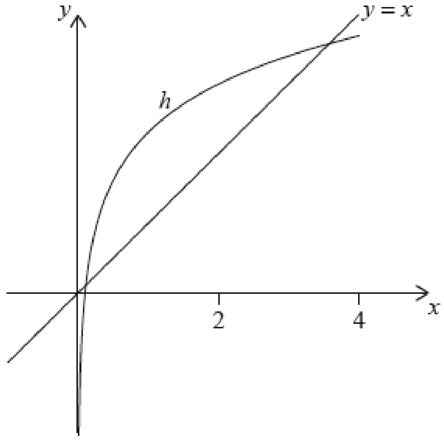
$$\text{a horizontal stretch of scale factor} q \text{followed by}$$
$$\text{a translation of} \begin{pmatrix} h \\ k \end{pmatrix}.$$

5a. Write down the value of q; *[1 mark]*

5b. Write down the value of h; *[1 mark]*

5c. Write down the value of k. *[1 mark]*

Let $h(x) = g(x) \times \cos(0.1x)$, for $0 < x < 4$. The following diagram shows the graph of h and the line $y = x$.



The graph of h intersects the graph of h^{-1} at two points. These points have x coordinates 0.111 and 3.31 correct to three significant figures.

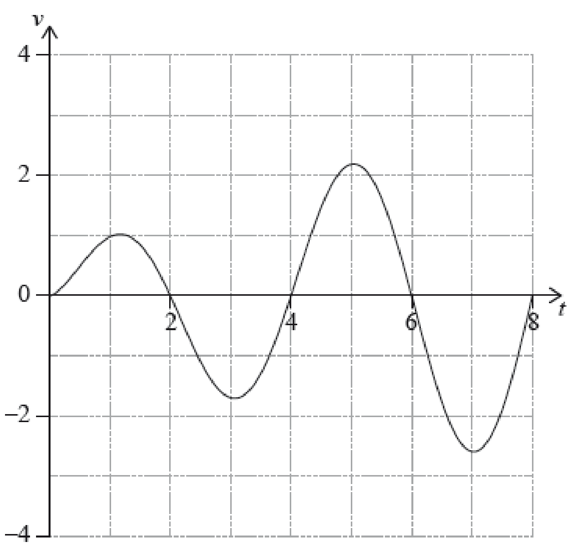
5d. Find $\int_{0.111}^{3.31} (h(x) - x)\,\mathrm{d}x$. *[2 marks]*

5e. Hence, find the area of the region enclosed by the graphs of h and h^{-1}. *[3 marks]*

5f. Let d be the vertical distance from a point on the graph of h to the line $y = x$. There is a point $\mathrm{P}(a, b)$ on the graph of h where d is a maximum. *[7 marks]*

Find the coordinates of P, where $0.111 < a < 3.31$.

A particle P moves along a straight line. Its velocity $v_{\mathrm{P}}\,\mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{P}} = \sqrt{t}\sin\left(\frac{\pi}{2}t\right)$, for $0 \leqslant t \leqslant 8$. The following diagram shows the graph of v_{P}.



6a. Write down the first value of t at which P changes direction. *[1 mark]*

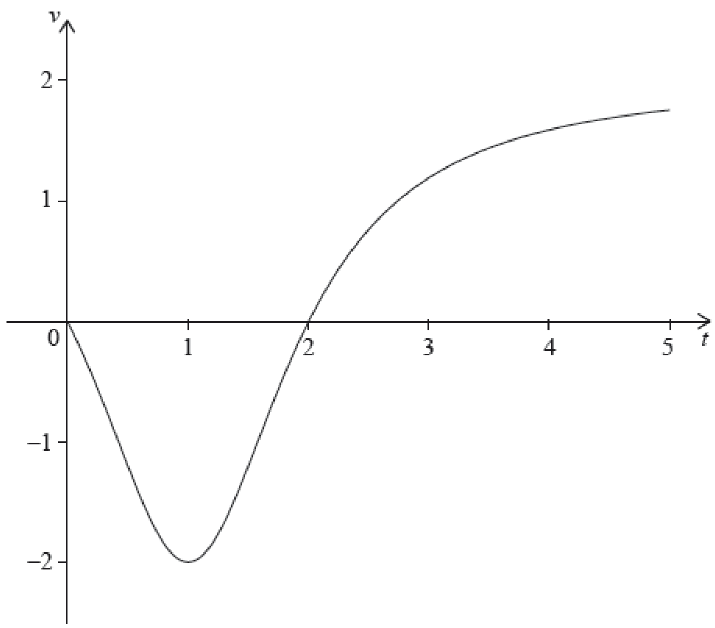
6b. Find the **total** distance travelled by P, for $0 \leqslant t \leqslant 8$. *[2 marks]*

6c. A second particle Q also moves along a straight line. Its velocity, $v_{\mathrm{Q}}\,\mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{Q}} = \sqrt{t}$ for $0 \leqslant t \leqslant 8$. After k seconds Q has travelled the same total distance as P. *[4 marks]*

Find k.

7. **Note: In this question, distance is in metres and time is in seconds.** *[6 marks]*

A particle moves along a horizontal line starting at a fixed point A. The velocity v of the particle, at time t, is given by $v(t) = \frac{2t^2-4t}{t^2-2t+2}$, for $0 \leqslant t \leqslant 5$. The following diagram shows the graph of v



There are t-intercepts at $(0,\ 0)$ and $(2,\ 0)$.

Find the maximum distance of the particle from A during the time $0 \leqslant t \leqslant 5$ and justify your answer.