# Definite Integrals *[113 marks]*

1. The derivative of the function $f$ is given by $f'(x) = \frac{6x}{x^2+1}$. *[5 marks]*

   The graph of $y = f(x)$ passes through the point $(1,\ 5)$. Find an expression for $f(x)$.

## Markscheme

recognizing need to integrate ***(M1)***

$\int \frac{6x}{x^2+1}\,\mathrm{d}x$ OR $u = x^2 + 1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\,\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\,\mathrm{d}x$ ***(A1)***

$= 3\ln\left(x^2 + 1\right)(+c)$ or $3\ln u(+c)$ ***A1***

correct substitution of $x = 1$ and $f(x) = 5$ or $x = 1$ and $u = 2$ into equation

using **their** integrated expression (must involve $c$) ***(M1)***

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2 + 1\right) + 5 - 3\ln 2$ $\left(= 3\ln\left(x^2 + 1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$

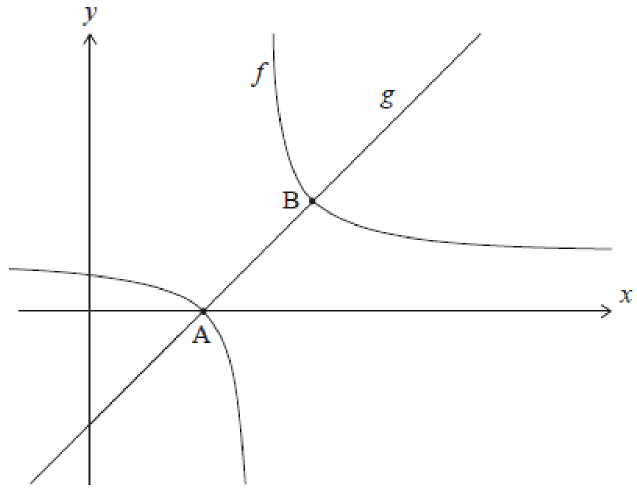
(or equivalent) ***A1***

**Note:** Accept the use of the modulus sign in working and the final answer.

***[5 marks]***

Consider the functions $f(x)=\frac{1}{x-4}+1$, for $x \neq 4$, and $g(x)=x-3$ for $x \in \mathbb{R}$.

The following diagram shows the graphs of $f$ and $g$.




The graphs of $f$ and $g$ intersect at points $\mathrm{A}$ and $\mathrm{B}$. The coordinates of $\mathrm{A}$ are $(3,\ 0)$.

2a. Find the coordinates of $\mathrm{B}$. *[5 marks]*

## Markscheme

$\frac{1}{x-4}+1=x-3$ ***(M1)***

$x^2-8x+15=0$ OR $(x-4)^2=1$ ***(A1)***

valid attempt to solve **their** quadratic ***(M1)***

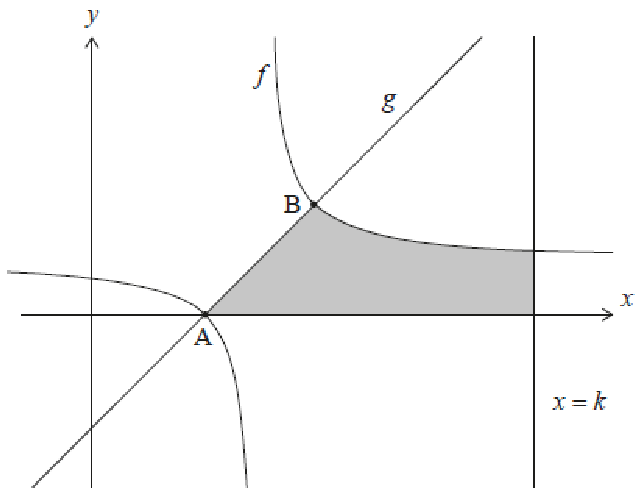
$(x-3)(x-5)=0$ OR $x=\frac{8 \pm \sqrt{8^2-4(1)(15)}}{2(1)}$ OR $(x-4)=\pm 1$

$x=5$ $(x=3,\ x=5)$ (may be seen in answer) ***A1***

$\mathrm{B}(5,\ 2)$ (accept $x=5,\ y=2$) ***A1***

***[5 marks]***

In the following diagram, the shaded region is enclosed by the graph of $f$, the graph of $g$, the $x$-axis, and the line $x = k$, where $k \in \mathbb{Z}$.




The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

2b. Find the value of $k$ and the value of $p$. *[10 marks]*

# Markscheme

recognizing two correct regions from $x=3$ to $x=5$ and from $x=5$ to $x=k$ ***(R1)***

triangle $+\int_5^k f(x)\,\mathrm{d}x$ OR $\int_3^5 g(x)\,\mathrm{d}x+\int_5^k f(x)\,\mathrm{d}x$ OR
$\int_3^5 (x-3)\,\mathrm{d}x+\int_5^k \left(\frac{1}{x-4}+1\right)\mathrm{d}x$

area of triangle is $2$ OR $\frac{2\cdot 2}{2}$ OR $\left(\frac{5^2}{2}-3(5)\right)-\left(\frac{3^2}{2}-3(3)\right)$ ***(A1)***

correct integration ***(A1)(A1)***

$\int\left(\frac{1}{x-4}+1\right)\mathrm{d}x=\ln(x-4)+x\ (+C)$

**Note:** Award ***A1*** for $\ln(x-4)$ and ***A1*** for $x$.
**Note:** The first three ***A*** marks may be awarded independently of the ***R*** mark.

substitution of **their** limits (for $x$) into **their** integrated function (in terms of $x$) ***(M1)***

$\ln(k-4)+k-(\ln 1+5)$

$[\ln(x-4)+x]_5^k=\ln(k-4)+k-5$ ***A1***

adding **their** two areas (in terms of $k$) and equating to $\ln p+8$ ***(M1)***

$2+\ln(k-4)+k-5=\ln p+8$

equating **their** non-log terms to $8$ (equation must be in terms of $k$) ***(M1)***

$k-3=8$

$k=11$ ***A1***

$11-4=p$

$p=7$ ***A1***

***[10 marks]***

3. Find the value of $\int_1^9\left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right)\mathrm{d}x$. *[5 marks]*

# Markscheme

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\,\mathrm{d}\,x = \int\left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}\,x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\,\mathrm{d}\,x = 3x-10x^{\frac{1}{2}}(+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

***[5 marks]***

Consider $f(x)=\frac{2x-4}{x^2-1}, -1<x<1$.

4a. Find $f'(x)$. *[2 marks]*

# Markscheme

attempt to use quotient rule (or equivalent) ***(M1)***

$f'(x)=\frac{\left(x^2-1\right)(2)-(2x-4)(2x)}{\left(x^2-1\right)^2}$ ***A1***

$=\frac{-2x^2+8x-2}{\left(x^2-1\right)^2}$

***[2 marks]***

4b. Show that, if $f'(x)=0$, then $x=2-\sqrt{3}$. *[3 marks]*

# Markscheme

$f'(x) = 0$

simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation ***A1***

**EITHER**

use of quadratic formula

$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$ ***A1***

**OR**

use of completing the square

$(x - 2)^2 = 3$ ***A1***

**THEN**

$x = 2 - \sqrt{3}$ (since $2 + \sqrt{3}$ is outside the domain) ***AG***

**Note:** Do not condone verification that $x = 2 - \sqrt{3} \Rightarrow f'(x) = 0$.

Do not award the final ***A1*** as follow through from part (i).

***[3 marks]***

For the graph of $y = f(x)$,

4c. find the coordinates of the $y$-intercept. *[1 mark]*

# Markscheme

(0, 4) ***A1***

***[1 mark]***

4d. show that there are no $x$-intercepts. *[2 marks]*

# Markscheme

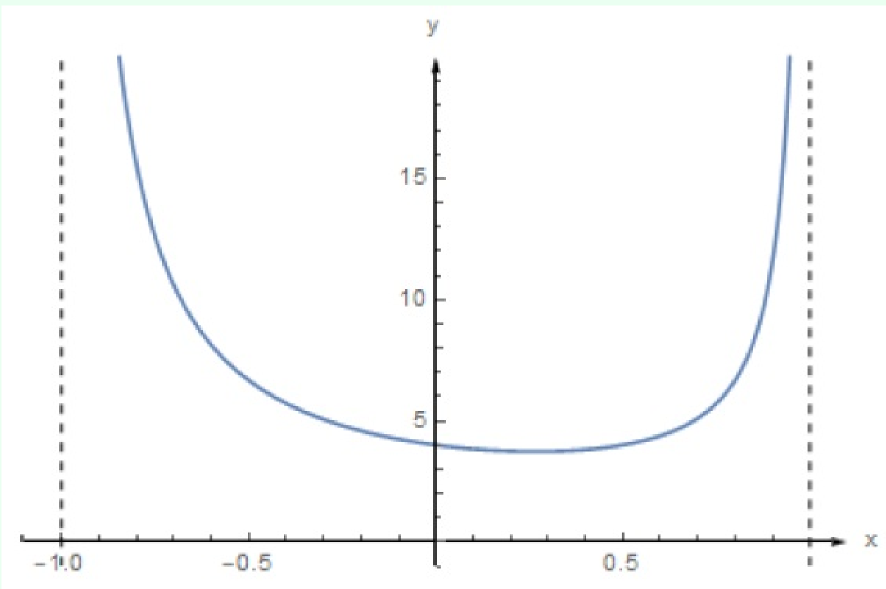
$2x - 4 = 0 \Rightarrow x = 2$ ***A1***

outside the domain ***R1***

***[2 marks]***

4e. sketch the graph, showing clearly any asymptotic behaviour. *[2 marks]*

# Markscheme




***A1A1***

award ***A1*** for concave up curve over correct domain with one minimum point in the first quadrant
award ***A1*** for approaching $x = \pm 1$ asymptotically

***[2 marks]***

4f. Show that $\frac{3}{x+1} - \frac{1}{x-1} = \frac{2x-4}{x^2-1}$. *[2 marks]*

## Markscheme

valid attempt to combine fractions (using common denominator) ***M1***

$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$ ***A1***

$= \frac{3x-3-x-1}{x^2-1}$

$= \frac{2x-4}{x^2-1}$ ***AG***

***[2 marks]***

4g. The area enclosed by the graph of $y = f\left(x\right)$ and the line $y = 4$ can be expressed as $\ln v$. Find the value of $v$. *[7 marks]*

# Markscheme

$f(x) = 4 \Rightarrow 2x - 4 = 4x^2 - 4$ ***M1***

($x = 0$ or) $x = \frac{1}{2}$ ***A1***

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$= \int_0^{\frac{1}{2}} \frac{3}{x+1} - \frac{1}{x-1}\mathrm{d}x$

**Note:** Ignore absence of, or incorrect limits up to this point.

$= [3\ln|x+1| - \ln|x-1|]_0^{\frac{1}{2}}$ ***A1***

$= 3\ln\frac{3}{2} - \ln\frac{1}{2}(-0)$

$= \ln\frac{27}{4}$ ***A1***

area is $2 - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ or $\int_0^{\frac{1}{2}} 4\,\mathrm{d}x - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$= 2 - \ln\frac{27}{4}$

$= \ln\frac{4\mathrm{e}^2}{27}$ ***A1***

$\left(\Rightarrow v = \frac{4\mathrm{e}^2}{27}\right)$

***[7 marks]***

5. Given that $\int_0^{\ln k} \mathrm{e}^{2x}\mathrm{d}x = 12$, find the value of $k$. *[6 marks]*

# Markscheme

$\frac{1}{2}\mathrm{e}^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^{\ln k} = \frac{1}{2}\mathrm{e}^{2\ln k} - \frac{1}{2}\mathrm{e}^0\right)$ ***(M1)***

$= \frac{1}{2}\mathrm{e}^{\ln k^2} - \frac{1}{2}\mathrm{e}^0$ ***(A1)***

Setting their equation $= 12$ ***M1***

**Note:** their equation must be an integrated expression with limits substituted.

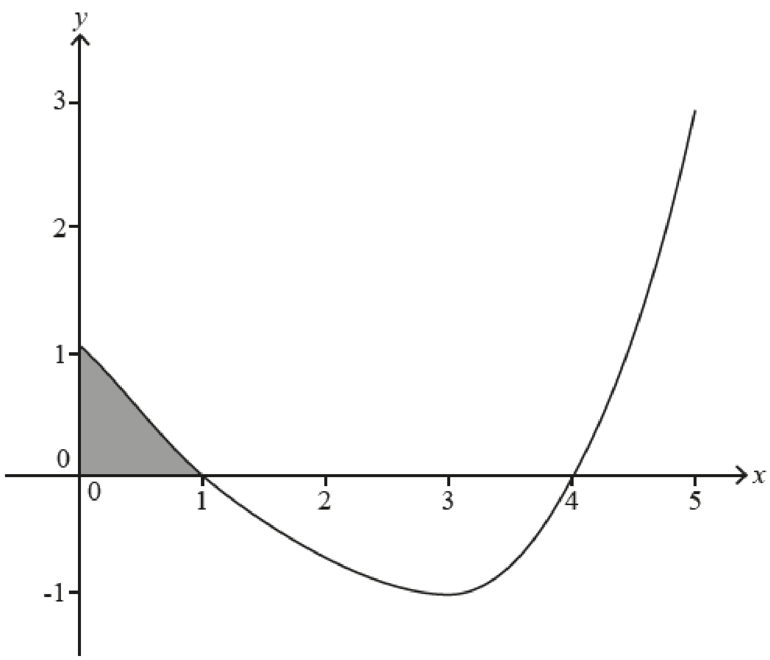
$\frac{1}{2}k^2 - \frac{1}{2} = 12$ ***A1***

$\left(k^2 = 25 \Rightarrow\right) k = 5$ ***A1***

**Note:** Do not award final ***A1*** for $k = \pm 5$.

***[6 marks]***

The graph of $y = f'(x)$, 0 ≤ $x$ ≤ 5 is shown in the following diagram. The curve intercepts the $x$-axis at (1, 0) and (4, 0) and has a local minimum at (3, −1).




6a. Write down the $x$-coordinate of the point of inflexion on the graph of $y = f(x)$. *[1 mark]*

## Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

3 ***A1***

***[1 mark]***

The shaded area enclosed by the curve $y = f'(x)$, the $x$-axis and the $y$-axis is 0.5. Given that $f(0) = 3$,

6b. find the value of $f(1)$. *[3 marks]*

## Markscheme

attempt to use definite integral of $f'(x)$ ***(M1)***

$\int_0^1 f'(x)\,\mathrm{d}x = 0.5$

$f(1) - f(0) = 0.5$ ***(A1)***

$f(1) = 0.5 + 3$

= 3.5 ***A1***

***[3 marks]***

The area enclosed by the curve $y = f'(x)$ and the $x$-axis between $x = 1$ and $x = 4$ is 2.5 .

6c. find the value of $f(4)$. *[2 marks]*

## Markscheme

$\int_1^4 f'(x)\,\mathrm{d}x = -2.5$ ***(A1)***

**Note:** ***(A1)*** is for −2.5.

$f(4) - f(1) = -2.5$

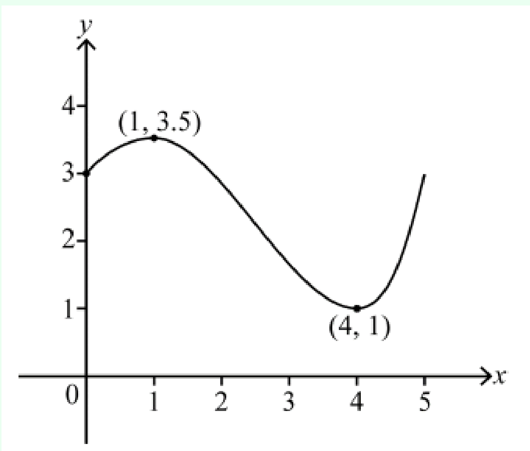
$f(4) = 3.5 - 2.5$

= 1 ***A1***

***[2 marks]***

6d. Sketch the curve $y = f(x)$, 0 ≤ $x$ ≤ 5 indicating clearly the coordinates of the maximum and minimum points and any intercepts with the coordinate axes. *[3 marks]*

## Markscheme



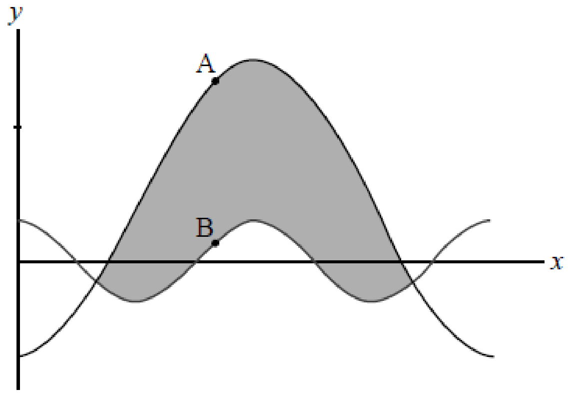

***A1A1A1***

***A1*** for correct shape over approximately the correct domain
***A1*** for maximum and minimum (coordinates or horizontal lines from 3.5 and 1 are required),
***A1*** for $y$-intercept at 3

***[3 marks]***

Consider the functions $f$ and $g$ defined on the domain $0 < x < 2\pi$ by $f(x) = 3\cos 2x$ and $g(x) = 4 - 11\cos x$.

The following diagram shows the graphs of $y = f(x)$ and $y = g(x)$




7a. Find the $x$-coordinates of the points of intersection of the two graphs. *[6 marks]*

# Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$3\cos 2x = 4 - 11\cos x$

attempt to form a quadratic in $\cos x$ ***M1***

$3\left(2\cos^2 x - 1\right) = 4 - 11\cos x$ ***A1***

$\left(6\cos^2 x + 11\cos x - 7 = 0\right)$

valid attempt to solve their quadratic ***M1***

$(3\cos x + 7)(2\cos x - 1) = 0$

$\cos x = \frac{1}{2}$ ***A1***

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ ***A1A1***

**Note:** Ignore any “extra” solutions.

***[6 marks]***

7b. Find the exact area of the shaded region, giving your answer in the form $p\pi + q\sqrt{3}$, where $p, q \in \mathbb{Q}$. *[5 marks]*

# Markscheme

consider (±) $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (4 - 11\cos x - 3\cos 2x)\, dx$ ***M1***

$= (\pm)\left[4x - 11\sin x - \frac{3}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$ ***A1***

**Note:** Ignore lack of or incorrect limits at this stage.

attempt to substitute their limits into their integral ***M1***

$= \frac{20\pi}{3} - 11\sin\frac{5\pi}{3} - \frac{3}{2}\sin\frac{10\pi}{3} - \left(\frac{4\pi}{3} - 11\sin\frac{\pi}{3} - \frac{3}{2}\sin\frac{2\pi}{3}\right)$

$= \frac{16\pi}{3} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$

$= \frac{16\pi}{3} + \frac{25\sqrt{3}}{2}$ ***A1A1***

***[5 marks]***

7c. At the points A and B on the diagram, the gradients of the two graphs are equal. *[6 marks]*

Determine the $y$-coordinate of A on the graph of $g$.

# Markscheme

attempt to differentiate both functions and equate ***M1***

$-6\sin 2x = 11\sin x$ ***A1***

attempt to solve for $x$ ***M1***

$11\sin x + 12\sin x\cos x = 0$

$\sin x\,(11 + 12\cos x) = 0$

$\cos x = -\frac{11}{12}$ (or $\sin x = 0$) ***A1***

$\Rightarrow y = 4 - 11\left(-\frac{11}{12}\right)$ ***M1***

$y = \frac{169}{12}\left(= 14\frac{1}{12}\right)$ ***A1***

***[6 marks]***

Given that $\int_{-2}^{2} f(x)\,\mathrm{d}x = 10$ and $\int_{0}^{2} f(x)\,\mathrm{d}x = 12$, find

8a. $\int_{-2}^{0} (f(x)\ + 2)\,\mathrm{d}x$. *[4 marks]*

# Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\int_{-2}^{0} f(x)\,\mathrm{d}x = 10 - 12 = -2$ ***(M1)(A1)***

$\int_{-2}^{0} 2\mathrm{d}x = [2x]_{-2}^{0} = 4$ ***A1***

$\int_{-2}^{0} (f(x)\ + 2)\,\mathrm{d}x = 2$ ***A1***

***[4 marks]***

8b. $\int_{-2}^{0} f(x + 2)\,\mathrm{d}x$. *[2 marks]*

# Markscheme

$\int_{-2}^{0} f(x + 2)\,\mathrm{d}x = \int_{0}^{2} f(x)\,\mathrm{d}x$ ***(M1)***

= 12 ***A1***

***[2 marks]***

Let $y = \arccos\left(\frac{x}{2}\right)$

9a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

## Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$y = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \left(= -\frac{1}{\sqrt{4-x^2}}\right)$ ***M1A1***

**Note:** ***M1*** is for use of the chain rule.

***[2 marks]***

9b. Find $\int_0^1 \arccos\left(\frac{x}{2}\right) \mathrm{d}x$. *[7 marks]*

## Markscheme

attempt at integration by parts ***M1***

$u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{4-x^2}}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$ ***(A1)***

$\int_0^1 \arccos\left(\frac{x}{2}\right) \mathrm{d}x = \left[x \arccos\left(\frac{x}{2}\right)\right]_0^1 + \int_0^1 \frac{1}{\sqrt{4-x^2}} dx$ ***A1***

using integration by substitution or inspection ***(M1)***

$\left[x \arccos\left(\frac{x}{2}\right)\right]_0^1 + \left[-\left(4-x^2\right)^{\frac{1}{2}}\right]_0^1$ ***A1***

**Note:** Award ***A1*** for $-\left(4-x^2\right)^{\frac{1}{2}}$ or equivalent.

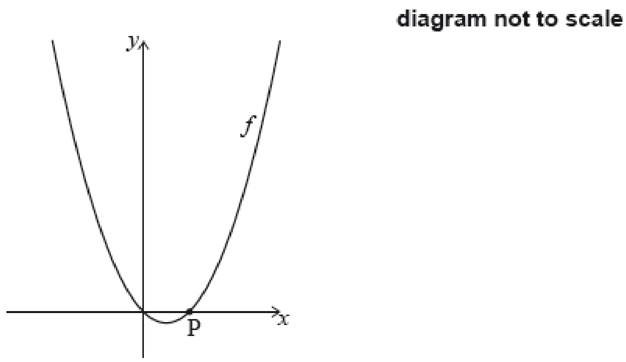
**Note:** Condone lack of limits to this point.

attempt to substitute limits into their integral ***M1***

$= \frac{\pi}{3} - \sqrt{3} + 2$ ***A1***

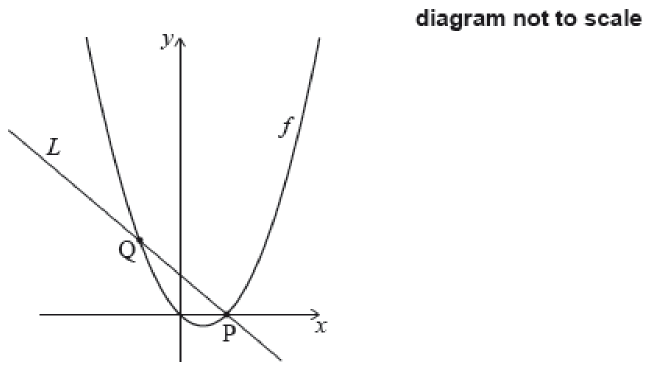
***[7 marks]***

Let $f(x) = x^2 - x$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of $f$.




The graph of $f$ crosses the $x$-axis at the origin and at the point $\mathrm{P}(1, 0)$.

The line $L$ intersects the graph of $f$ at another point Q, as shown in the following diagram.




10. Find the area of the region enclosed by the graph of $f$ and the line $L$. *[6 marks]*

# Markscheme

valid approach *(M1)*

*eg* $\int L - f,\ \int_{-1}^{1}(1-x^2)\mathrm{d}x$, splitting area into triangles and integrals

correct integration *(A1)(A1)*

*eg* $\left[x - \frac{x^3}{3}\right]_{-1}^{1},\ -\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^2}{2} + x$

substituting **their** limits into **their** integrated function and subtracting (in any order) *(M1)*

*eg* $1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)$

**Note:** Award ***M0*** for substituting into original or differentiated function.

area $= \frac{4}{3}$ ***A2*** ***N3***

***[6 marks]***

11a. Express $x^2 + 3x + 2$ in the form $(x+h)^2 + k$. *[1 mark]*

# Markscheme

$x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

***[1 mark]***

11b. Factorize $x^2 + 3x + 2$. *[1 mark]*

# Markscheme

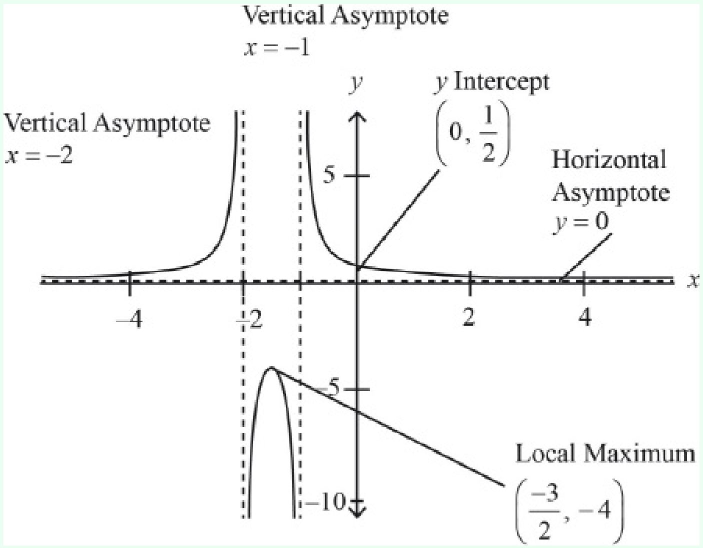
$x^2 + 3x + 2 = (x+2)(x+1)$ ***A1***

***[1 mark]***

Consider the function $f(x)=\frac{1}{x^2+3x+2}, x\in\mathbb{R}, x\neq -2, x\neq -1$.

11c. Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the $y$-intercept and the local maximum. *[5 marks]*

# Markscheme




***A1*** for the shape

***A1*** for the equation $y=0$

***A1*** for asymptotes $x=-2$ and $x=-1$

***A1*** for coordinates $\left(-\frac{3}{2},\ -4\right)$

***A1*** $y$-intercept $\left(0,\ \frac{1}{2}\right)$

***[5 marks]***

11d. Hence find the value of $p$ if $\int_0^1 f(x)\mathrm{d}x=\ln(p)$. *[4 marks]*

# Markscheme

$$\int_0^1 \frac{1}{x+1} - \frac{1}{x+2}\mathrm{d}x$$

$= [\ln(x+1) - \ln(x+2)]_0^1$ ***A1***

$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

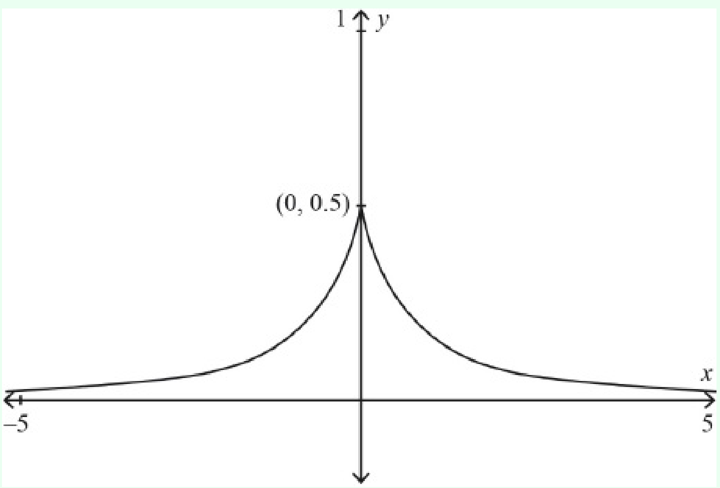
$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore p = \frac{4}{3}$

***[4 marks]***

11e. Sketch the graph of $y = f(|x|)$. *[2 marks]*

# Markscheme




symmetry about the $y$-axis ***M1***

correct shape ***A1***

**Note:** Allow ***FT*** from part (b).

***[2 marks]***

11f. Determine the area of the region enclosed between the graph of $y = f(|x|)$, the $x$-axis and the lines with equations $x = -1$ and $x = 1$. *[3 marks]*

## Markscheme

$2\int_0^1 f(x)\mathrm{d}x$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

**Note:** Do not award ***FT*** from part (e).

***[3 marks]***