Diff equations *[84 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2+y^2-xy}{x^2}$, with $y = 2$ when $x = 1$.

1a. Use Euler's method, with step length $h = 0.1$, to find an approximate value of y when $x = 1.4$. *[5 marks]*

Markscheme

x	y	$\frac{dy}{dx}$
1	2	6
1.1	2.6	7.22
1.2	3.32	8.89652
1.3	4.21	11.26
1.4	5.34	

(M1)(A1)(A1)(A1)A1

$y(1.4) \approx 5.34$

Note: Award ***A1*** for each correct y value.
For the intermediate y values, accept answers that are accurate to 2 significant figures.
The final y value must be accurate to 3 significant figures or better.

[5 marks]

1b. Express $m^2 - 2m + 4$ in the form $(m-a)^2 + b$, where a,$b \in \mathbb{Z}$. *[1 mark]*

Markscheme

$m^2 - 2m + 4 = (m-1)^2 + 3 \quad (a = 1,\ b = 3)$ ***A1***

[1 mark]

1c. Solve the differential equation, for $x > 0$, giving your answer in the form $y = f(x)$. *[10 marks]*

Markscheme

recognition of homogeneous equation,
let $y = vx$ ***M1***

the equation can be written as

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 4 + v^2 - v$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2v + 4$

$\int \frac{1}{v^2-2v+4}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***M1***

Note: Award ***M1*** for attempt to separate the variables.

$\int \frac{1}{(v-1)^2+3}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ from part (c)(i) ***M1***

$\frac{1}{\sqrt{3}}\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \ln x \ \ (+c)$ ***A1A1***

$x = 1, y = 2 \Rightarrow v = 2$

$\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) = \ln 1 + c$ ***M1***

Note: Award ***M1*** for using initial conditions to find c.

$\Rightarrow c = \frac{\pi}{6\sqrt{3}}\ (= 0.302)$ ***A1***

$\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \sqrt{3}\ln x + \frac{\pi}{6}$

substituting $v = \frac{y}{x}$ ***M1***

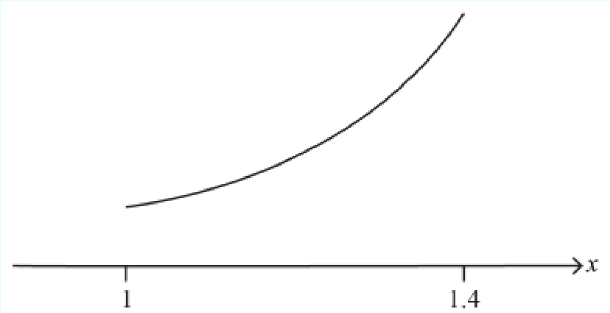
Note: This ***M1*** may be awarded earlier.

$y = x\left(\sqrt{3}\tan\left(\sqrt{3}\ln x + \frac{\pi}{6}\right) + 1\right)$ ***A1***

[10 marks]

1d. Sketch the graph of $y = f\left(x\right)$ for $1 \leqslant x \leqslant 1.4$. *[1 mark]*

Markscheme



curve drawn over correct domain ***A1***

[1 mark]

1e. With reference to the curvature of your sketch in part (c)(iii), and without further calculation, explain whether you conjecture $f\left(1.4\right)$ will be less than, equal to, or greater than your answer in part (a). *[2 marks]*

Markscheme

the sketch shows that f is concave up ***A1***

Note: Accept f' is increasing.

this means the tangent drawn using Euler's method will give an underestimate of the real value, so $f\left(1.4\right)$ > estimate in part (a) ***R1***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$, where $x \ne 0$.

2a. Given that $y\left(1\right) = 1$, use Euler's method with step length h = 0.25 to find an approximation for $y\left(2\right)$. Give your answer to two significant figures. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to apply Euler's method ***(M1)***

$$x_{n+1} = x_n + 0.25;\ y_{n+1} = y_n + 0.25 \times \left(1 + \frac{y_n}{x_n}\right)$$

x	y	$\frac{dy}{dx}$
1.00	1.00000	2.00000
1.25	1.50000	2.20000
1.50	2.05000	2.36667
1.75	2.64167	2.50952
2.00	3.26905	

(A1)(A1)

Note: Award ***A1*** for correct x values, **A1** for first three correct y values.

y = 3.3 ***A1***

[4 marks]

2b. Solve the equation $\frac{dy}{dx} = 1 + \frac{y}{x}$ for $y(1) = 1$. *[6 marks]*

Markscheme

METHOD 1

$I\left(x\right) = \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***(M1)***

$= \mathrm{e}^{-\ln x}$

$= \frac{1}{x}$ ***(A1)***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \frac{1}{x}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \frac{1}{x}$

$\frac{y}{x} = \ln\left|x\right| + C$ ***A1***

$y\left(1\right) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln\left|x\right| + x$ ***A1***

METHOD 2

$v = \frac{y}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y$ ***(A1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v$ ***M1***

$\int 1\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$

$v = \ln\left|x\right| + C$

$\frac{y}{x} = \ln\left|x\right| + C$ ***A1***

$y\left(1\right) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln\left|x\right| + x$ ***A1***

[6 marks]

2c. Find the percentage error when $y\left(2\right)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. *[3 marks]*

Markscheme

$y\left(2\right) = 2\ln 2 + 2 = 3.38629\ldots$

percentage error $= \frac{3.38629\ldots - 3.3}{3.38629\ldots} \times 100\%$ ***(M1)(A1)***

= 2.5% ***A1***

[3 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

3a. Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

integrating factor $= \mathrm{e}^{\int \frac{x}{x^2+1}\mathrm{d}x}$ ***(M1)***

$\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\ln(x^2+1)$ ***(M1)***

Note: Award ***M1*** for use of $u = x^2+1$ for example or $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x)$.

integrating factor $= \mathrm{e}^{\frac{1}{2}\ln(x^2+1)}$ ***A1***

$= \mathrm{e}^{\ln\left(\sqrt{x^2+1}\right)}$ ***A1***

Note: Award ***A1*** for $\mathrm{e}^{\ln\sqrt{u}}$ where $u = x^2+1$.

$= \sqrt{x^2+1}$ ***AG***

METHOD 2

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}y$ ***M1A1***

$\sqrt{x^2+1}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express in the form $\sqrt{x^2+1} \times (\text{LHS of de})$.

so $\sqrt{x^2+1}$ is an integrating factor for this differential equation ***AG***

[4 marks]

3b. Solve the differential equation giving your answer in the form $y = f(x)$. *[6 marks]*

Markscheme

$\sqrt{x^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{\sqrt{x^2+1}}y = x\sqrt{x^2+1}$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right) = x\sqrt{x^2+1}$

$y\sqrt{x^2+1} = \int x\sqrt{x^2+1}\mathrm{d}x \left(y = \frac{1}{\sqrt{x^2+1}}\int x\sqrt{x^2+1}\mathrm{d}x\right)$ ***A1***

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$ ***(M1)A1***

Note: Award ***M1*** for using an appropriate substitution.

Note: Condone the absence of C.

substituting $x = 0, y = 1 \Rightarrow C = \frac{2}{3}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = \frac{1}{3}(x^2+1) + \frac{2}{3\sqrt{x^2+1}} \left(y = \frac{(x^2+1)^{\frac{3}{2}}+2}{3\sqrt{x^2+1}}\right)$ ***A1***

[6 marks]

4a. Consider the differential equation *[3 marks]*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right),\ x > 0.$$

Use the substitution $y = vx$ to show that the general solution of this differential equation is

$$\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + \text{Constant}.$$

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***M1***

the differential equation becomes

$v + x\frac{dv}{dx} = f(v)$ ***A1***

$\int \frac{dv}{f(v)-v} = \int \frac{dv}{x}$ ***A1***

integrating, Constant $\int \frac{dv}{f(v)-v} = \ln x + \text{ Constant}$ ***AG***

[3 marks]

4b. Hence, or otherwise, solve the differential equation *[10 marks]*

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}, x > 0,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = g(x)$.

Markscheme

EITHER

$f(v) = 1 + 3v + v^2$ ***(A1)***

$\left(\int \frac{\mathrm{d}v}{f(v)-v} =\right)\ \int \frac{\mathrm{d}v}{1+3v+v^2-v} = \ln x + C$ ***M1A1***

$\int \frac{\mathrm{d}v}{(1+v)^2} = (\ln x + C)$ ***A1***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v}\ \ (= \ln x + C)$ ***A1***

OR

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 3v + v^2$ ***A1***

$\int \frac{\mathrm{d}v}{1+2v+v^2} = \int \frac{1}{x}\mathrm{d}x$ ***M1***

$\int \frac{\mathrm{d}v}{(1+v)^2}\ \ \left(= \int \frac{1}{x}\mathrm{d}x\right)$ ***(A1)***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v} = \ln x (+C)$ ***A1A1***

THEN

substitute $y = 1$ or $v = 1$ when $x = 1$ (***M1)***

therefore $C = -\frac{1}{2}$ ***A1***

Note: This ***A1*** can be awarded anywhere in their solution.

substituting for v,

$-\frac{1}{\left(1+\frac{y}{x}\right)} = \ln x - \frac{1}{2}$ ***M1***

Note: Award for correct substitution of $\frac{y}{x}$ into their expression.

$1 + \frac{y}{x} = \frac{1}{\frac{1}{2} - \ln x}$ ***(A1)***

Note: Award for any rearrangement of a correct expression that has y in the numerator.

$y = x\left(\frac{1}{\left(\frac{1}{2}-\ln x\right)} - 1\right)\ \ (\text{or equivalent})$ ***A1***

$\left(= x\left(\frac{1+2\ln x}{1-2\ln x}\right)\right)$

[10 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right)y = x^2$, given that $y = 2$ when $x = 0$.

5a. Show that $1 + x^2$ is an integrating factor for this differential equation. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

attempting to find an integrating factor ***(M1)***

$\int \frac{2x}{1+x^2}\mathrm{d}x = \ln(1 + x^2)$ ***(M1)A1***

IF is $\mathrm{e}^{\ln(1+x^2)}$ ***(M1)A1***

$= 1 + x^2$ ***AG***

METHOD 2

multiply by the integrating factor

$(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x^2(1 + x^2)$ ***M1A1***

left hand side is equal to the derivative of $(1 + x^2)y$

A3

[5 marks]

5b. Hence solve this differential equation. Give the answer in the form $y = f(x)$. *[6 marks]*

Markscheme

$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=(1+x^2)x^2$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x^2)y\right]=x^2+x^4$

$(1+x^2)y=\left(\int x^2+x^4\mathrm{d}x=\right)\frac{x^3}{3}+\frac{x^5}{5}(+c)$ ***A1A1***

$y=\frac{1}{1+x^2}\left(\frac{x^3}{3}+\frac{x^5}{5}+c\right)$

$x=0, y=2\Rightarrow c=2$ ***M1A1***

$y=\frac{1}{1+x^2}\left(\frac{x^3}{3}+\frac{x^5}{5}+2\right)$ ***A1***

[6 marks]

Consider the expression $\frac{1}{\sqrt{1+ax}}-\sqrt{1-x}$ where $a\in\mathbb{Q},\ a\neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx+bx^2$, where $b\in\mathbb{Q}$.

6a. Find the value of a and the value of b. *[6 marks]*

Markscheme

attempt to expand binomial with negative fractional power ***(M1)***

$\frac{1}{\sqrt{1+ax}}=(1+ax)^{-\frac{1}{2}}=1-\frac{ax}{2}+\frac{3a^2x^2}{8}+\ldots$ ***A1***

$\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1-\frac{x}{2}-\frac{x^2}{8}+\ldots$ ***A1***

$\frac{1}{\sqrt{1+ax}}-\sqrt{1-x}=\frac{(1-a)}{2}x+\left(\frac{3a^2+1}{8}\right)x^2+\ldots$

attempt to equate coefficients of x or x^2 ***(M1)***

$x\ :\ \ \frac{1-a}{2}=4b;\ \ x^2\ :\ \frac{3a^2+1}{8}=b$

attempt to solve simultaneously ***(M1)***

$a=-\frac{1}{3},\ b=\frac{1}{6}$ ***A1***

[6 marks]

6b. State the restriction which must be placed on x for this expansion to be valid. *[1 mark]*

Markscheme

$|x| < 1$ ***A1***

[1 mark]

The function f is defined by $f(x) = \arcsin(2x)$, where $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

7a. By finding a suitable number of derivatives of f, find the first two non-zero terms in the Maclaurin series for f. *[8 marks]*

Markscheme

$f(x) = \arcsin(2x)$

$f'(x) = \frac{2}{\sqrt{1-4x^2}}$ ***M1A1***

Note: Award ***M1A0*** for $f'(x) = \frac{1}{\sqrt{1-4x^2}}$

$f''(x) = \frac{8x}{(1-4x^2)^{\frac{3}{2}}}$ ***A1***

EITHER

$$f'''(x) = \frac{8(1-4x^2)^{\frac{3}{2}} - 8x\left(\frac{3}{2}(-8x)(1-4x^2)^{\frac{1}{2}}\right)}{(1-4x^2)^3} \quad \left(= \frac{8(1-4x^2)^{\frac{3}{2}} + 96x^2(1-4x^2)^{\frac{1}{2}}}{(1-4x^2)^3}\right)$$ ***A1***

OR

$$f'''(x) = 8(1-4x^2)^{-\frac{3}{2}} + 8x\left(-\frac{3}{2}(1-4x^2)^{-\frac{5}{2}}\right)(-8x) \quad \left(= 8(1-4x^2)^{-\frac{3}{2}} + 96x\right.$$

A1

THEN

substitute $x = 0$ into f or any of its derivatives ***(M1)***

$f(0) = 0$, $f'(0) = 2$ and $f''(0) = 0$ ***A1***

$f'''(0) = 8$

the Maclaurin series is

$f(x) = 2x + \frac{8x^3}{6} + \ldots \left(= 2x + \frac{4x^3}{3} + \ldots\right)$ ***(M1)A1***

[8 marks]

7b. Hence or otherwise, find $\lim\limits_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3}$. *[3 marks]*

Markscheme

METHOD 1

$$\lim_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3} = \lim_{x \to 0} \frac{2x + \frac{4x^3}{3} + \ldots - 2x}{8x^3}$$ ***M1***

$$= \lim_{x \to 0} \frac{\frac{4}{3} + \ldots \text{terms with} x}{8}$$ ***(M1)***

$$= \frac{1}{6}$$ ***A1***

Note: Condone the omission of +… in their working.

METHOD 2

$$\lim_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3} = \frac{0}{0}$$ indeterminate form, using L'Hôpital's rule

$$= \lim_{x \to 0} \frac{\frac{2}{\sqrt{1-4x^2}} - 2}{24x^2}$$ ***M1***

$$= \frac{0}{0}$$ indeterminate form, using L'Hôpital's rule again

$$= \lim_{x \to 0} \frac{\frac{8x}{\left(1-4x^2\right)^{\frac{3}{2}}}}{48x} \left(= \lim_{x \to 0} \frac{1}{6\left(1-4x^2\right)^{\frac{3}{2}}} \right)$$ ***M1***

Note: Award ***M1*** only if their previous expression is in indeterminate form.

$$= \frac{1}{6}$$ ***A1***

Note: Award ***FT*** for use of their derivatives from part (a).

[3 marks]