Diff equations *[84 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2+y^2-xy}{x^2}$, with $y = 2$ when $x = 1$.

1a. Use Euler's method, with step length $h = 0.1$, to find an approximate value of y when $x = 1.4$. *[5 marks]*

1b. Express $m^2 - 2m + 4$ in the form $(m-a)^2 + b$, where $a,b \in \mathbb{Z}$. *[1 mark]*

1c. Solve the differential equation, for $x > 0$, giving your answer in the form $y = f(x)$. *[10 marks]*

1d. Sketch the graph of $y = f(x)$ for $1 \leqslant x \leqslant 1.4$. *[1 mark]*

1e. With reference to the curvature of your sketch in part (c)(iii), and without further calculation, explain whether you conjecture $f(1.4)$ will be less than, equal to, or greater than your answer in part (a). *[2 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$, where $x \neq 0$.

2a. Given that $y(1) = 1$, use Euler's method with step length h = 0.25 to find an approximation for $y(2)$. Give your answer to two significant figures. *[4 marks]*

2b. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$ for $y(1) = 1$. *[6 marks]*

2c. Find the percentage error when $y(2)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. *[3 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

3a. Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. *[4 marks]*

3b. Solve the differential equation giving your answer in the form $y = f(x)$. *[6 marks]*

4a. Consider the differential equation *[3 marks]*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right),\ x > 0.$$

Use the substitution $y = vx$ to show that the general solution of this differential equation is

$$\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + \text{Constant}.$$

4b. Hence, or otherwise, solve the differential equation *[10 marks]*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3xy + y^2}{x^2}, x > 0,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = g(x)$.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right) y = x^2$, given that $y = 2$ when $x = 0$.

5a. Show that $1 + x^2$ is an integrating factor for this differential equation. *[5 marks]*

5b. Hence solve this differential equation. Give the answer in the form $y = f(x)$. *[6 marks]*

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q},\ a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

6a. Find the value of a and the value of b. *[6 marks]*

6b. State the restriction which must be placed on x for this expansion to be valid. *[1 mark]*

The function f is defined by $f(x) = \arcsin(2x)$, where $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

7a. By finding a suitable number of derivatives of f, find the first two non-zero terms in the Maclaurin series for f. *[8 marks]*

7b. Hence or otherwise, find $\lim_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3}$. *[3 marks]*