Volumes 1 *[43 marks]*

Consider the function $f(x) = \sqrt{x^2 - 1}$, where $1 \leq x \leq 2$.

1a. Sketch the curve $y = f(x)$, clearly indicating the coordinates of the endpoints. *[2 marks]*

1b. Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$. *[3 marks]*

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1c. State the domain and range of f^{-1}. *[2 marks]*

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The curve $y = f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

1d. Show that the volume, $V\ \mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V = \pi\left(\frac{1}{3}h^3 + h\right)$. *[3 marks]*

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1e. Hence, determine the maximum volume of the container. *[2 marks]*

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At $t = 0$, the container is empty. Water is then added to the container at a constant rate of $0.4 \mathrm{~m}^3 \mathrm{~s}^{-1}$.

1f. Find the time it takes to fill the container to its maximum volume. *[2 marks]*

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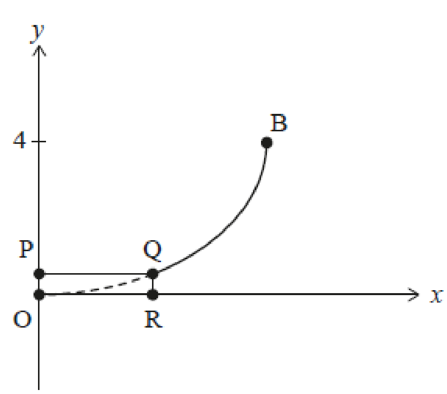
1g. Find the rate of change of the height of the water when the container is filled to half its maximum volume. *[6 marks]*

2. *[5 marks]*

The following diagram shows the curve $\frac{x^2}{36} + \frac{(y-4)^2}{16} = 1$, where $h \leq y \leq 4$.

diagram not to scale



The curve from point Q to point B is rotated 360° about the y-axis to form the interior surface of a bowl. The rectangle OPQR, of height h cm, is rotated 360° about the y-axis to form a solid base.

The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\ \mathrm{cm}^3$, determine the height of the base.

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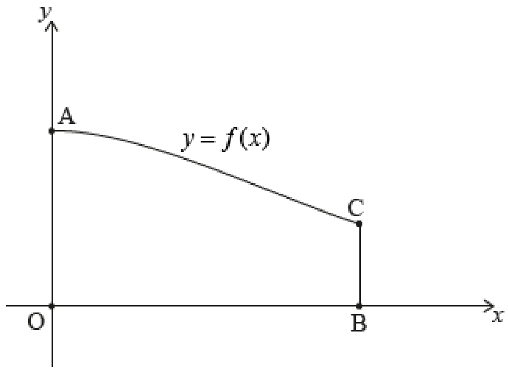
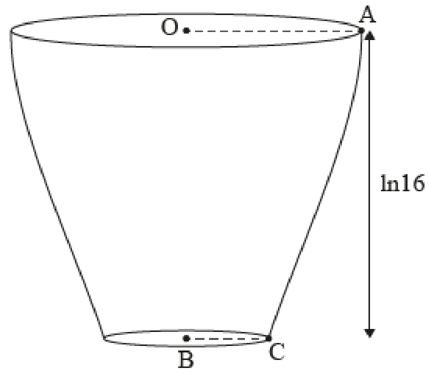
A function f is defined by $f(x)=\frac{ke^{\frac{x}{2}}}{1+e^{x}}$, where $x \in \mathbb{R}, x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis and the line $x = \ln 16$ is rotated 360° about the x-axis to form a solid of revolution.

3a. Show that the volume of the solid formed is $\frac{15k^{2}\pi}{34}$ cubic units. *[6 marks]*

Pedro wants to make a small bowl with a volume of $300\ \mathrm{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.





The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

3b. Find the value of k that satisfies the requirements of Pedro's design. *[2 marks]*

3c. Find OA. *[2 marks]*

3d. Find BC. *[2 marks]*

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

3e. By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly. *[4 marks]*

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3f. State the cross-sectional radius of the bowl at this point. *[2 marks]*