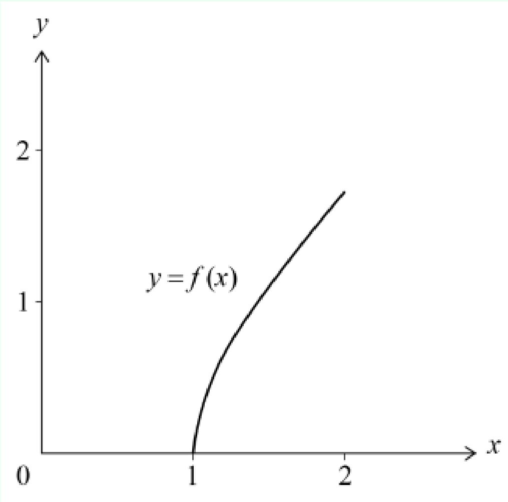
Volumes 1 *[43 marks]*

Consider the function $f(x) = \sqrt{x^2 - 1}$, where $1 \le x \le 2$.

1a. Sketch the curve $y = f(x)$, clearly indicating the coordinates of the endpoints. *[2 marks]*

Markscheme



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1, 0)$ and $\left(2, \sqrt{3}\right) (= (2, 1.73))$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

[2 marks]

1b. Show that the inverse function of f is given by $f^{-1}(x) = \sqrt{x^2 + 1}$. *[3 marks]*

Markscheme

interchanging x and y (seen anywhere) ***M1***

$x=\sqrt{y^2-1}$

$x^2=y^2-1$ ***A1***

$y=\sqrt{x^2+1}$ ***A1***

$f^{-1}(x)=\sqrt{x^2+1}$ ***AG***

[3 marks]

1c. State the domain and range of f^{-1}. *[2 marks]*

Markscheme

$0\le x\le\sqrt{3}$ OR domain $\left[0,\sqrt{3}\right](=[0,1.73])$ ***A1***

$1\le y\le 2$ OR $1\le f^{-1}(x)\le 2$ OR range $[1,2]$ ***A1***

[2 marks]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

1d. Show that the volume, $V\ \mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$. *[3 marks]*

Markscheme

attempt to substitute $x = \sqrt{y^2 + 1}$ into the correct volume formula ***(M1)***

$V = \pi \int_0^h \left(\sqrt{y^2+1}\right)^2 \mathrm{d}y \left(= \pi \int_0^h \left(y^2+1\right) \mathrm{d}y\right)$ ***A1***

$= \pi\left[\frac{1}{3}y^3 + y\right]_0^h$ ***A1***

$= \pi\left(\frac{1}{3}h^3 + h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g. $\pi \int_0^h \left(\sqrt{x^2+1}\right)^2 \mathrm{d}x$

[3 marks]

1e. Hence, determine the maximum volume of the container. *[2 marks]*

Markscheme

attempt to substitute $h = \sqrt{3}\ (= 1.732\ldots)$ into V ***(M1)***

$V = 10.8828\ldots$

$V = 10.9\left(\mathrm{m}^3\right)\ \left(= 2\sqrt{3}\pi\right)\left(\mathrm{m}^3\right)$ ***A1***

[2 marks]

At $t = 0$, the container is empty. Water is then added to the container at a constant rate of $0.4\ \mathrm{m}^3\ \mathrm{s}^{-1}$.

1f. Find the time it takes to fill the container to its maximum volume. *[2 marks]*

Markscheme

time $= \frac{10.8828...}{0.4}\left(=\frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$= 27.207...$

$= 27.2\left(=5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

1g. Find the rate of change of the height of the water when the container is filled to half its maximum volume. *[6 marks]*

Markscheme

attempt to find the height of the tank when $V = 5.4414...\ \left(=\sqrt{3}\pi\right)$ ***(M1)***

$\pi\left(\frac{1}{3}h^3+h\right)=5.4414...\ \left(=\sqrt{3}\pi\right)$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V=\pi\left(\frac{1}{3}h^3+h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{\pi(h^2+1)}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t}=\pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t}=0.4$ ***(M1)***

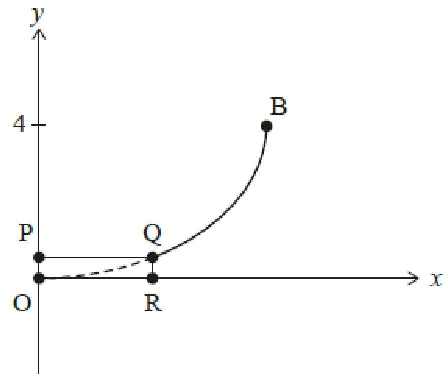
$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{0.4}{\pi(1.1818...^2+1)}=0.053124...$

$=0.0531\left(\mathrm{m\,s^{-1}}\right)$ ***A1***

[6 marks]

2. The following diagram shows the curve $\frac{x^2}{36}+\frac{(y-4)^2}{16}=1$, where $h \le y \le 4$. *[5 marks]*

diagram not to scale



The curve from point Q to point B is rotated 360° about the y-axis to form the interior surface of a bowl. The rectangle OPQR, of height h cm, is rotated 360° about the y-axis to form a solid base.

The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\ \mathrm{cm}^3$, determine the height of the base.

Markscheme

attempts to express x^2 in terms of y ***(M1)***

$V=\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y=285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48}-\frac{h^2}{4}+\frac{8}{3}\right)=285$ or equivalent for h.

$h=0.7926\ldots$
$h=0.793$ (cm) ***A2***

[5 marks]

A function f is defined by $f(x)=\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}$, where $x\in\mathbb{R}, x\geq 0$ and $k\in\mathbb{R}^+$.

The region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis and the line $x=\ln 16$ is rotated 360° about the x-axis to form a solid of revolution.

3a. Show that the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units. *[6 marks]*

Markscheme

attempt to use $V=\pi\int_a^b (f(x))^2\,\mathrm{d}x$ ***(M1)***

$V=\pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2\mathrm{d}x\left(V=k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{(1+\mathrm{e}^x)^2}\,\mathrm{d}x\right)$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+e^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1 + e^x \Rightarrow du = e^x\,dx$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0, u = 2$ and when $x = \ln 16, u = 17$

$V = k^2\pi\int_2^{17} \frac{1}{u^2}\,du$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = e^x \Rightarrow du = e^x\,dx$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0, u = 1$ and when $x = \ln 16, u = 16$

$V = k^2\pi\int_1^{16} \frac{1}{(1+u)^2}\,du$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

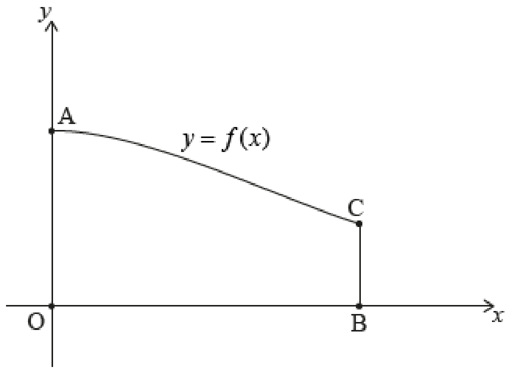
$= k^2\pi\left(\frac{1}{2} - \frac{1}{17}\right)$ ***A1***

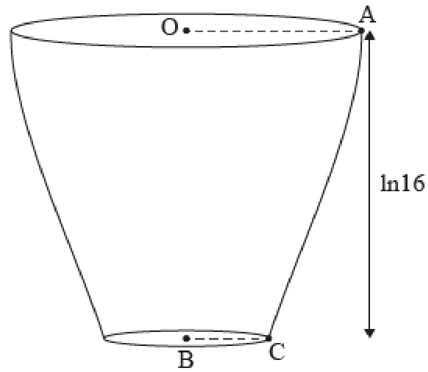
so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

Pedro wants to make a small bowl with a volume of $300\ \mathrm{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.





The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

3b. Find the value of k that satisfies the requirements of Pedro's design. *[2 marks]*

Markscheme

a valid algebraic or graphical attempt to find k ***(M1)***

$k^2 = \frac{300\times 34}{15\pi}$

$k = 14.7\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}}\right)$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

3c. Find OA. *[2 marks]*

Markscheme

attempting to find $\mathrm{OA} = f(0) = \frac{k}{2}$

with $k = 14.712\ldots\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}}\right)$ ***(M1)***

$\mathrm{OA} = 7.36\left(= \sqrt{\frac{170}{\pi}}\right)$ ***A1***

[2 marks]

3d. Find BC. *[2 marks]*

Markscheme

attempting to find $\mathrm{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712\ldots\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}}\right)$ ***(M1)***

$\mathrm{BC} = 3.46\left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}}\right)$ ***A1***

[2 marks]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

3e. By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly. *[4 marks]*

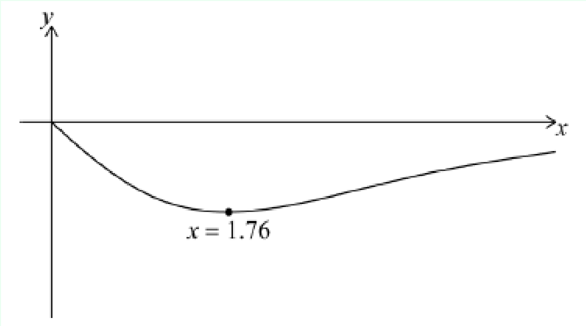
Markscheme

EITHER

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award **M1** for attempting to use quotient rule or product rule

differentiation. $f'(x) = \frac{k\mathrm{e}^{\frac{x}{2}}(1-\mathrm{e}^{x})}{2(1+\mathrm{e}^{x})^{2}}$



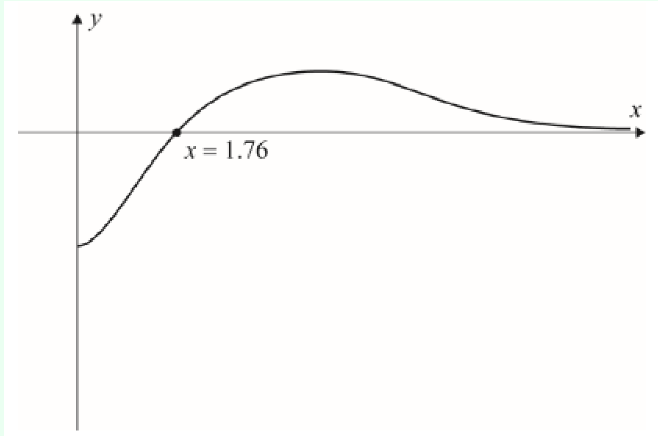
for $x > 0$ graph decreasing to the local minimum ***A1***

before increasing towards the x-axis ***A1***

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation. $f''(x) = \frac{ke^{\frac{x}{2}}\left(e^{2x} - 6e^{x} + 1\right)}{4\left(1 + e^{x}\right)^{3}}$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76\left(= \ln\left(2\sqrt{2} + 3\right)\right)$ ***A1***

Note: Only award ***A*** marks if either graph is seen.

[4 marks]

3f. State the cross-sectional radius of the bowl at this point. *[2 marks]*

Markscheme

attempting to find $f(1.76\ldots)$ ***(M1)***

the cross-sectional radius at this point is $5.20\left(\sqrt{\frac{85}{\pi}}\right)(\text{cm})$ ***A1***

[2 marks]