Volumes 2 *[98 marks]*

The function f is defined by $f(x) = \mathrm{e}^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

1a. Find $f'(x)$. *[3 marks]*

Markscheme

attempt to use product rule ***(M1)***

$f'(x) = 3\mathrm{e}^{2x} + 2\mathrm{e}^{2x}(3x - 4)\left(= \mathrm{e}^{2x}(6x - 5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3\mathrm{e}^{2x}$, $6x\mathrm{e}^{2x}$ and $-8\mathrm{e}^{2x}$ seen or implied.

[3 marks]

1b. Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. *[3 marks]*

Markscheme

$f'(x) = 1$ ***(M1)***

$x = 0.86299\ldots$

$x = 0.863$ ***A1***

$y = -7.92719\ldots$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

1c. Find the volume of this solid. *[4 marks]*

Markscheme

x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$$\pi \int_0^{\frac{4}{3}} \left(\mathrm{e}^{2x}(3x-4)\right)^2 \mathrm{d}\,x$$ ***(A1)***

Note: This ***(A1)*** can be awarded if the $\mathrm{d}x$ is omitted.

$= 164.849\ldots$
$= 165$ ***A1***

[4 marks]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

Find the value of

1d. $(f \circ g)(0)$. *[2 marks]*

Markscheme

attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905\ldots$

$= -7.39\left(= -\mathrm{e}^2\right)$ ***A1***

[2 marks]

1e. $(f \circ g)'(0)$. *[3 marks]*

Markscheme

attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905\ldots)$ ***(A1)***

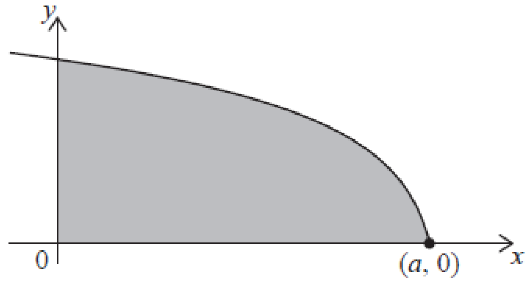
$= 14.7781\ldots$

$= 14.8\left(= 2\mathrm{e}^2\right)$ ***A1***

[3 marks]

Let $f(x)=\sqrt{12-2x}, x \leq a$. The following diagram shows part of the graph of f.

The shaded region is enclosed by the graph of f, the x-axis and the y-axis.



The graph of f intersects the x-axis at the point $(a, 0)$.

2a. Find the value of a. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognize $f(x)=0$ ***(M1)***

eg $\sqrt{12-2x}=0,\ \ 2x=12$

$a=6$ (accept $x=6$, $(6,\ 0)$) ***A1 N2***

[2 marks]

2b. Find the volume of the solid formed when the shaded region is revolved 360° about the x-axis. *[5 marks]*

Markscheme

attempt to substitute either **their** limits or the function into volume formula (must involve f^2) ***(M1)***

eg $\int_0^6 f^2 \,\mathrm{d}x$, $\pi \int \left(\sqrt{12-2x}\right)^2$, $\pi \int_0^6 12 - 2x \,\mathrm{d}x$

correct integration of each term ***A1 A1***

eg $12x - x^2$, $12x - x^2 + c$, $\left[12x - x^2\right]_0^6$

substituting limits into **their integrated** function and subtracting (in any order) ***(M1)***

eg $\pi\left(12(6)-(6)^2\right)-\pi(0)$, $72\pi - 36\pi$, $\left(12(6)-(6)^2\right)-(0)$

Note: Award ***M0*** if candidate has substituted into f, f^2 or f'.

volume$=36\pi$ ***A1 N2***

[5 marks]

Consider the function defined by $f(x) = \frac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

3a. State the equation of the vertical asymptote on the graph of $y = f(x)$. *[1 mark]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x = k$ ***A1***

[1 mark]

3b. State the equation of the horizontal asymptote on the graph of $y = f(x)$. *[1 mark]*

Markscheme

$y = k$ ***A1***

[1 mark]

3c. Use an algebraic method to determine whether f is a self-inverse function. *[4 marks]*

Markscheme

METHOD 1

$(f\circ f)(x)=\frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$ ***M1***

$=\frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$ ***A1***

$=\frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$

$=\frac{k^2x-5x}{k^2-5}$ ***A1***

$=\frac{x(k^2-5)}{k^2-5}$

$=x$

$(f\circ f)(x)=x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x))=x$ could be seen anywhere in the candidate’s working to award ***R1***.

METHOD 2

$f(x)=\frac{kx-5}{x-k}$

$x=\frac{ky-5}{y-k}$ ***M1***

Note: Interchanging x and y can be done at any stage.

$x(y-k)=ky-5$ ***A1***

$xy-xk=ky-5$

$xy-ky=xk-5$

$y(x-k)=kx-5$ ***A1***

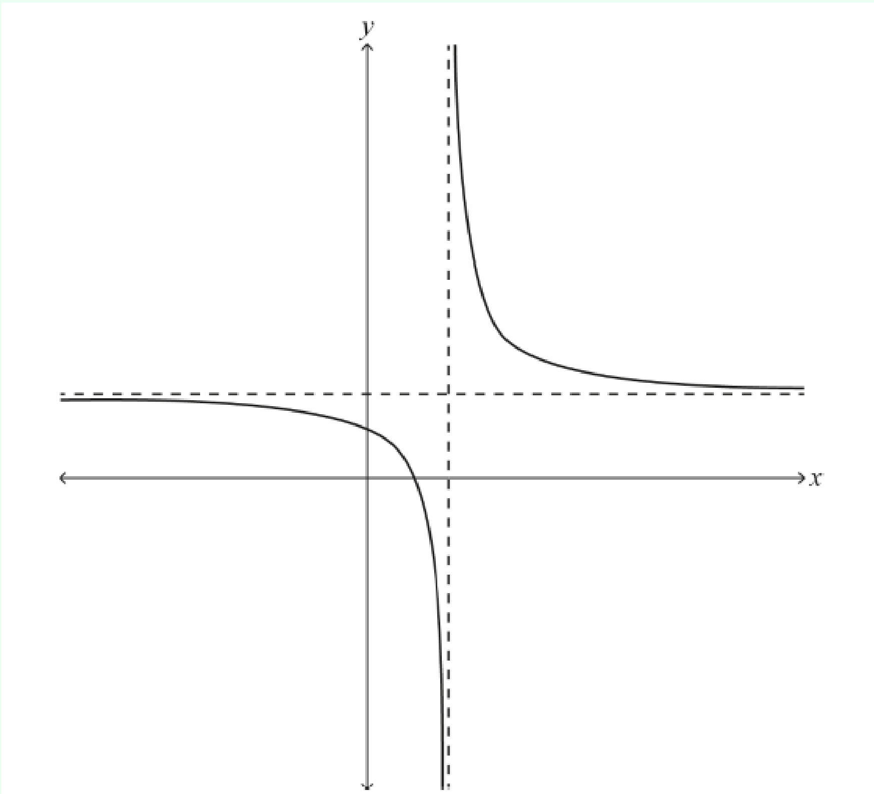
$y=f^{-1}(x)=\frac{kx-5}{x-k}$ (hence f is self-inverse) ***R1***

[4 marks]

Consider the case where $k = 3$.

3d. Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. *[3 marks]*

Markscheme



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0,\ \frac{5}{3}\right)$ and $\left(\frac{5}{3},\ 0\right)$ ***A1***

[3 marks]

3e. The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ \ln 2)$, where $a,\ b \in \mathbb{Z}$. *[6 marks]*

Markscheme

METHOD 1

volume $= \pi \int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p + \frac{q}{x-3}$ ***M1***

$\frac{3x-5}{x-3} = 3 + \frac{4}{x-3}$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24x-56}{(x-3)^2}$ ***A1***

THEN

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}$ ***A1***

volume $= \pi \int_5^7 \left(9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}\right) \mathrm{d}x$

$= \pi\left[9x + 24\ln(x-3) - \frac{16}{x-3}\right]_5^7$ ***A1***

$= \pi\lfloor(63 + 24\ln 4 - 4) - (45 + 24\ln 2 - 8)\rfloor$

$= \pi(22 + 24\ln 2)$ ***A1***

METHOD 2

volume $= \pi \int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

substituting $u = x - 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x - 5 = 3(u+3) - 5 = 3u + 4$

volume $= \pi \int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$ ***M1***

$= \pi \int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\, \mathrm{d}u$ ***A1***

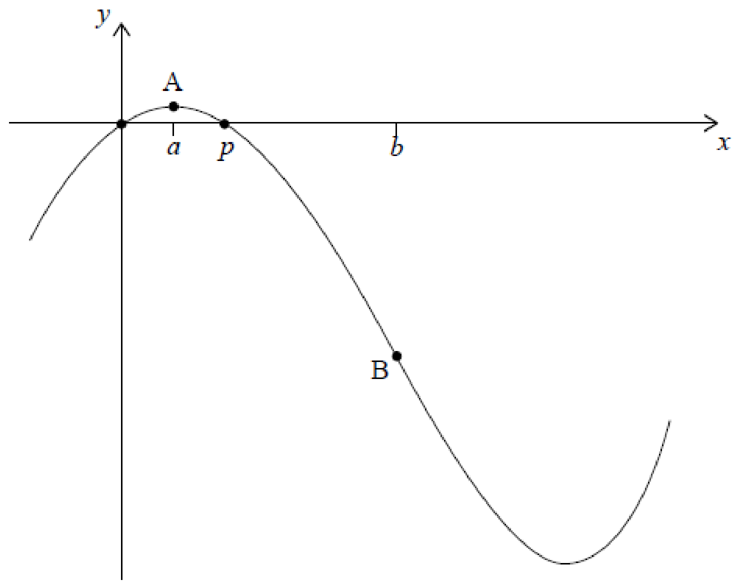
$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$= \pi(22 + 24\ln 2)$ ***A1***

[6 marks]

Let $f\left(x\right) = x^4 - 54x^2 + 60x$, for $-1 \leqslant x \leqslant 6$. The following diagram shows the graph of f.



There are x-intercepts at $x = 0$ and at $x = p$. There is a maximum at point A where $x = a$, and a point of inflexion at point B where $x = b$.

4a. Find the value of p. *[2 marks]*

Markscheme

evidence of valid approach ***(M1)***

eg $f\left(x\right) = 0,\ y = 0$

1.13843

$p = 1.14$ ***A1 N2***

[2 marks]

4b. Write down the coordinates of A. *[2 marks]*

Markscheme

$0.562134, \ 16.7641$

$(0.562, 16.8)$ ***A2 N2***

[2 marks]

4c. Find the equation of the tangent to the graph of f at A. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg tangent at maximum point is horizontal, $f' = 0$

$y = 16.8$ (must be an equation) ***A1 N2***

[2 marks]

4d. Find the coordinates of B. *[5 marks]*

Markscheme

METHOD 1 (using GDC)

valid approach ***M1***

eg $f'' = 0$, max/min on f', $x = -3$

sketch of either f' or f'', with max/min or root (respectively) ***(A1)***

$x = 3$ ***A1 N1***

substituting **their** x value into f ***(M1)***

eg $f(3)$

$y = -225$ (exact) (accept $(3, -225)$) ***A1 N1***

METHOD 2 (analytical)

$f'' = 12x^2 - 108$ ***A1***

valid approach ***(M1)***

eg $f'' = 0$, $x = \pm 3$

$x = 3$ ***A1 N1***

substituting **their** x value into f ***(M1)***

eg $f(3)$

$y = -225$ (exact) (accept $(3, -225)$) ***A1 N1***

[5 marks]

4e. Find the rate of change of f at B. *[2 marks]*

Markscheme

recognizing rate of change is f' ***(M1)***

eg y', $f'(3)$

rate of change is -156 (exact) ***A1 N2***

[2 marks]

4f. Let R be the region enclosed by the graph of f, the x-axis and the lines $x = p$ and $x = b$. The region R is rotated 360º about the x-axis. Find the volume of the solid formed. *[3 marks]*

Markscheme

attempt to substitute **either their** limits **or** the function into volume formula ***(M1)***

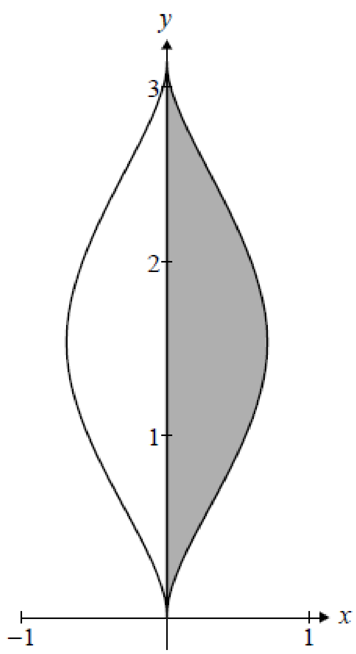
eg $\int_{1.14}^{3} f^2$, $\pi \int \left(x^4 - 54x^2 + 60x\right)^2 \mathrm{d}x$, $25\,752.0$

$80\,902.3$

volume $= 80\,900$ ***A2 N3***

[3 marks]

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \leqslant y \leqslant \pi$.



5a. Using implicit differentiation, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[4 marks]*

Markscheme

valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y \cos y \frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{3\sin^2 y \cos y}$ ***A1***

[4 marks]

5b. Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. *[4 marks]*

Markscheme

at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8}{3\sqrt{3}}(= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[4 marks]

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

5c. Find the area of R. *[3 marks]*

Markscheme

$x = \sqrt{\frac{1}{2}\sin^3 y}$ ***(M1)***

$\int_0^\pi \sqrt{\frac{1}{2}\sin^3 y \, \mathrm{d}y}$ ***(A1)***

$= 1.24$ ***A1***

[3 marks]

5d. The region R is now rotated about the y-axis, through 2π radians, to form a solid. *[6 marks]*

By writing $\sin^3 y$ as $\left(1-\cos^2 y\right)\sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$.

Markscheme

use of volume $= \int \pi x^2 \, \mathrm{d}y$ ***(M1)***

$= \int_0^\pi \frac{1}{2}\pi \sin^3 y \, \mathrm{d}y$ ***A1***

$= \frac{1}{2}\pi \int_0^\pi \left(\sin y - \sin y \cos^2 y\right) \mathrm{d}y$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate ***(M1)***

$= \frac{1}{2}\pi\left[-\cos y + \frac{1}{3}\cos^3 y\right]_0^\pi$ ***A1A1***

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$= \frac{1}{2}\pi\left(1-\frac{1}{3}\right) - \frac{1}{2}\pi\left(-1+\frac{1}{3}\right)$ ***A1***

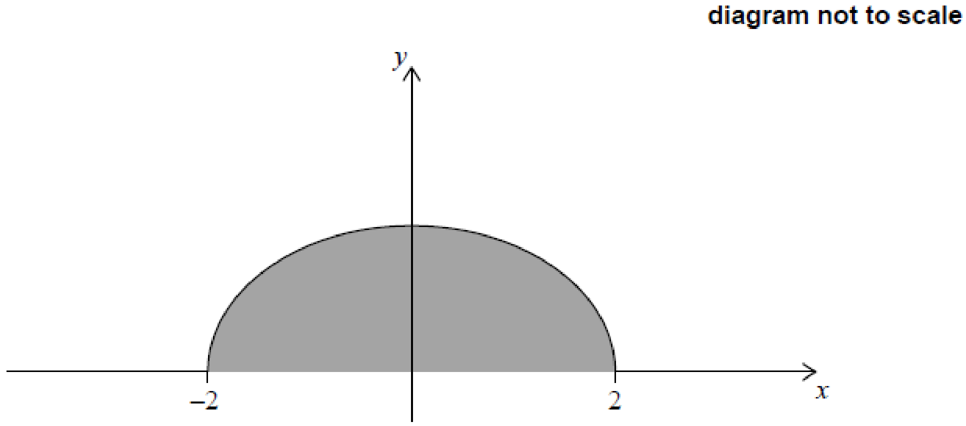
$= \frac{2\pi}{3}$ ***AG***

Note: Do not accept decimal answer equivalent to $\frac{2\pi}{3}$.

[6 marks]

All lengths in this question are in metres.

Consider the function $f(x) = \sqrt{\frac{4-x^2}{8}}$, for $-2 \le x \le 2$. In the following diagram, the shaded region is enclosed by the graph of f and the x-axis.



A container can be modelled by rotating this region by 360° about the x-axis.

6a. Find the volume of the container. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to substitute correct limits or the function into formula involving f^2 ***(M1)***

eg $\pi \int_{-2}^{2} y^2 \, \mathrm{d}y$, $\pi \int \left(\sqrt{\frac{4-x^2}{8}}\right)^2 \mathrm{d}x$

4.18879

volume = 4.19, $\frac{4}{3}\pi$ (exact) (m^3) ***A2 N3***

Note: If candidates have their GDC incorrectly set in degrees, award ***M*** marks where appropriate, but no ***A*** marks may be awarded. Answers from degrees are *p* = 13.1243 and *q* = 26.9768 in (b)(i) and 12.3130 or 28.3505 in (b)(ii).

[3 marks]

Water can flow in and out of the container.

The volume of water in the container is given by the function $g(t)$, for $0 \leq t \leq 4$, where t is measured in hours and $g(t)$ is measured in m^3. The rate of change of the volume of water in the container is given by $g'(t) = 0.9 - 2.5\cos(0.4t^2)$.

The volume of water in the container is increasing only when $p < t < q$.

6b. Find the value of p and of q. *[3 marks]*

Markscheme

recognizing the volume increases when g' is positive ***(M1)***

eg $g'(t) > 0$, sketch of graph of g' indicating correct interval

1.73387, 3.56393

$p = 1.73$, $p = 3.56$ ***A1A1 N3***

[3 marks]

6c. During the interval $p < t < q$, he volume of water in the container increases by $k\,m^3$. Find the value of k. *[3 marks]*

Markscheme

valid approach to find change in volume ***(M1)***

eg $g(q) - g(p)$, $\int_p^q g'(t)\,dt$

3.74541

total amount = 3.75 (m^3) ***A2 N3***

[3 marks]

6d. When $t = 0$, the volume of water in the container is $2.3\,m^3$. It is known that the container is never completely full of water during the 4 hour period. *[5 marks]*

Find the minimum volume of empty space in the container during the 4 hour period.

Markscheme

Note: There may be slight differences in the final answer, depending on which values candidates carry through from previous parts. Accept answers that are consistent with correct working.

recognizing when the volume of water is a maximum ***(M1)***

eg maximum when $t = q$, $\int_0^q g'(t)\,\mathrm{d}t$

valid approach to find maximum volume of water ***(M1)***

eg $2.3 + \int_0^q g'(t)\,\mathrm{d}t$, $2.3 + \int_0^p g'(t)\,\mathrm{d}t + 3.74541$, 3.85745

correct expression for the difference between volume of container and maximum value ***(A1)***

eg $4.18879 - \left(2.3 + \int_0^q g'(t)\,\mathrm{d}t\right)$, 4.19 − 3.85745

0.331334

0.331 (m^3) ***A2 N3***

[5 marks]

Let $f(x) = 6 - \ln(x^2 + 2)$, for $x \in \mathbb{R}$. The graph of f passes through the point $(p, 4)$, where $p > 0$.

7a. Find the value of p. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach ***(M1)***

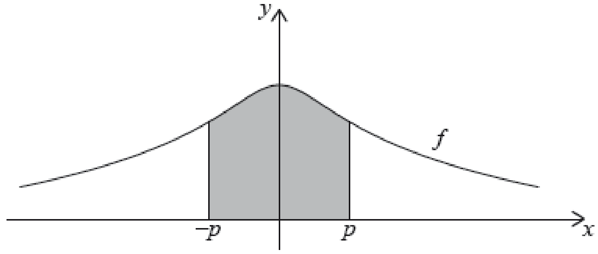
eg $f(p) = 4$, intersection with $y = 4$, ± 2.32

2.32143

$p = \sqrt{\mathrm{e}^2 - 2}$ (exact), 2.32 ***A1*** ***N2***

[2 marks]

7b. The following diagram shows part of the graph of f. *[3 marks]*



The region enclosed by the graph of f, the x-axis and the lines $x = -p$ and $x = p$ is rotated 360° about the x-axis. Find the volume of the solid formed.

Markscheme

attempt to substitute **either their** limits **or** the function into volume formula (must involve f^2, accept reversed limits and absence of π and/or $\mathrm{d}x$, but do not accept any other errors) ***(M1)***

eg $\int_{-2.32}^{2.32} f^2,\ \pi \int \left(6 - \ln(x^2+2)\right)^2 \mathrm{d}x,\ 105.675$

331.989

$\text{volume} = 332$ ***A2*** ***N3***

[3 marks]

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

8. Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geqslant 0$ and the y-axis for $y \geqslant 0$ is rotated through 2π about the x-axis. *[3 marks]*

Markscheme

Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 \mathrm{d}x$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) \mathrm{d}x$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark.

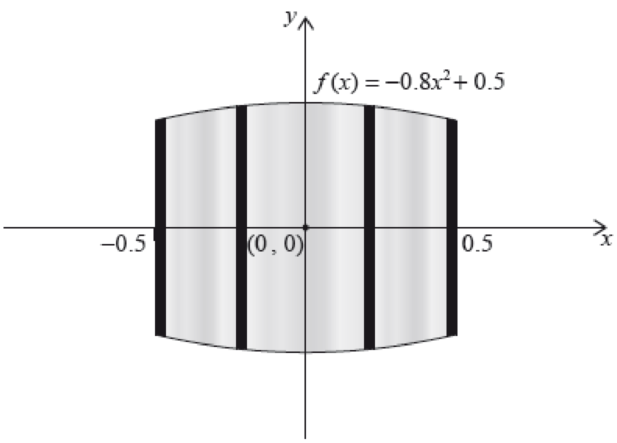
Do not condone absence of or multiples of π.

$= 19.4 \quad \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

All lengths in this question are in metres.

Let $f(x) = -0.8x^2 + 0.5$, for $-0.5 \leqslant x \leqslant 0.5$. Mark uses $f(x)$ as a model to create a barrel. The region enclosed by the graph of f, the x-axis, the line $x = -0.5$ and the line $x = 0.5$ is rotated 360° about the x-axis. This is shown in the following diagram.



9a. Use the model to find the volume of the barrel. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to substitute correct limits or the function into the formula involving y^2

eg $\pi\int_{-0.5}^{0.5} y^2\mathrm{d}x, \pi\int\left(-0.8x^2+0.5\right)^2\mathrm{d}x$

0.601091

volume $= 0.601(\mathrm{m}^3)$ ***A2 N3***

[3 marks]

9b. The empty barrel is being filled with water. The volume $V\mathrm{m}^3$ of water in the barrel after t minutes is given by $V = 0.8(1 - \mathrm{e}^{-0.1t})$. How long will it take for the barrel to be half-full? *[3 marks]*

Markscheme

attempt to equate half **their** volume to V ***(M1)***

eg $0.30055 = 0.8(1 - \mathrm{e}^{-0.1t})$, graph

4.71104

4.71 (minutes) ***A2 N3***

[3 marks]