Volumes 2 *[98 marks]*

The function f is defined by $f(x)= \mathrm{e}^{2x}(3x-4)$, where $x \in \mathbb{R}$.

1a. Find $f'(x)$. *[3 marks]*

1b. Hence or otherwise, find the coordinates of the point on the graph of $y=f(x)$ where the tangent is parallel to the line $y=x$. *[3 marks]*

The region enclosed by the curve $y=f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

1c. Find the volume of this solid. *[4 marks]*

Consider a function g such that $g(0)=1$ and $g'(0)=2$.

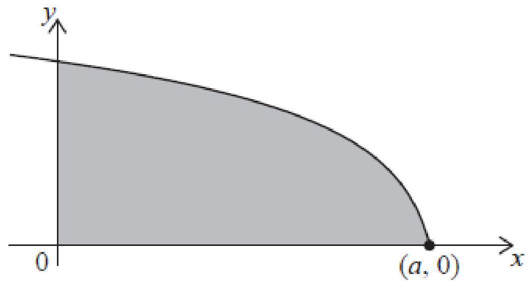
Find the value of

1d. $(f \circ g)(0)$. *[2 marks]*

1e. $(f \circ g)'(0)$. *[3 marks]*

Let $f(x) = \sqrt{12-2x}, x \leq a$. The following diagram shows part of the graph of f.

The shaded region is enclosed by the graph of f, the x-axis and the y-axis.



The graph of f intersects the x-axis at the point $(a, 0)$.

2a. Find the value of a. *[2 marks]*

2b. Find the volume of the solid formed when the shaded region is revolved 360° about the x-axis. *[5 marks]*

Consider the function defined by $f(x) = \frac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

3a. State the equation of the vertical asymptote on the graph of $y = f(x)$. *[1 mark]*

3b. State the equation of the horizontal asymptote on the graph of $y = f(x)$. *[1 mark]*

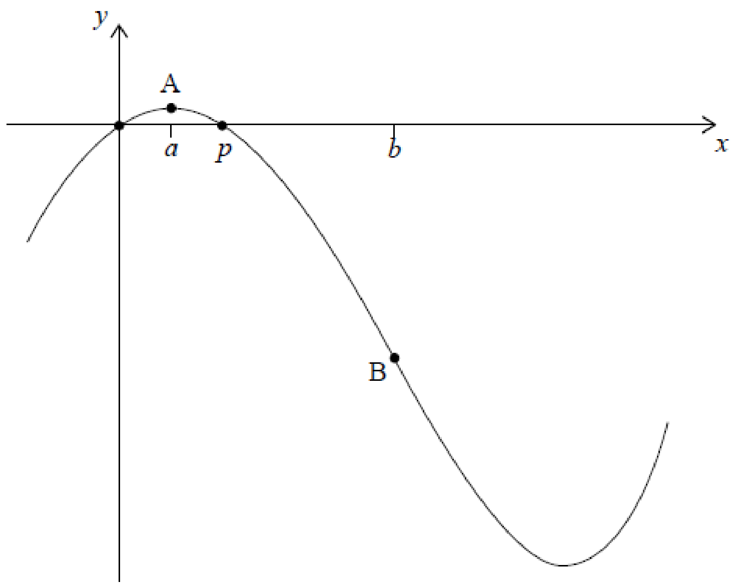
3c. Use an algebraic method to determine whether f is a self-inverse function. *[4 marks]*

Consider the case where $k = 3$.

3d. Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. *[3 marks]*

3e. The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ \ln 2)$, where $a,\ b \in \mathbb{Z}$. *[6 marks]*

Let $f(x) = x^4 - 54x^2 + 60x$, for $-1 \leqslant x \leqslant 6$. The following diagram shows the graph of f.



There are x-intercepts at $x = 0$ and at $x = p$. There is a maximum at point A where $x = a$, and a point of inflexion at point B where $x = b$.

4a. Find the value of p. *[2 marks]*

4b. Write down the coordinates of A. *[2 marks]*

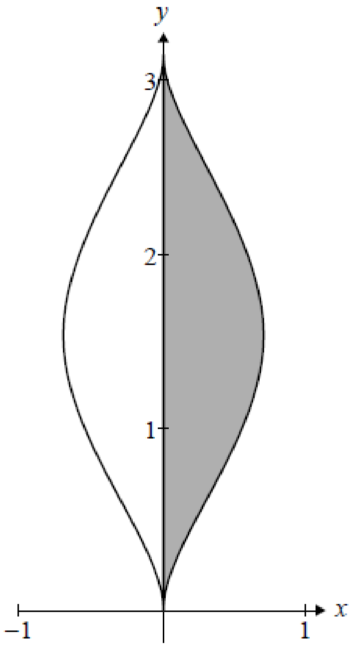
4c. Find the equation of the tangent to the graph of f at A. *[2 marks]*

4d. Find the coordinates of B. *[5 marks]*

4e. Find the rate of change of f at B. *[2 marks]*

4f. Let R be the region enclosed by the graph of f, the x-axis and the lines $x = p$ and $x = b$. The region R is rotated 360º about the x-axis. Find the volume of the solid formed. *[3 marks]*

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \leqslant y \leqslant \pi$.



5a. Using implicit differentiation, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[4 marks]*

5b. Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. *[4 marks]*

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

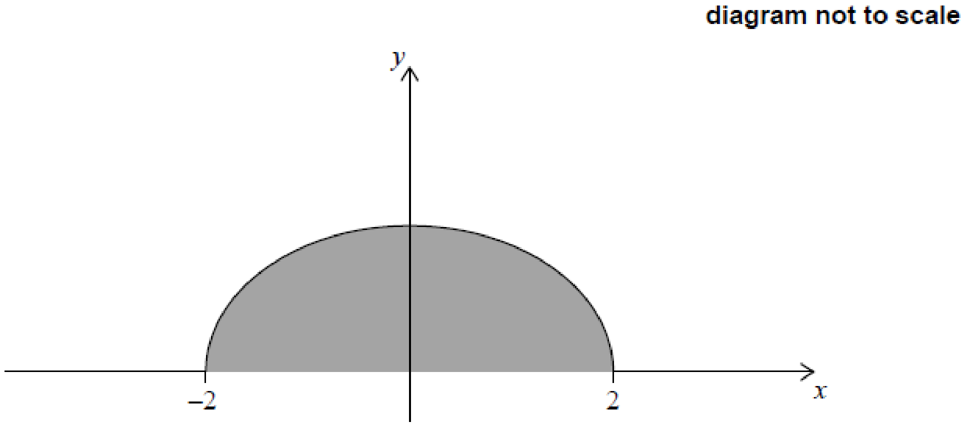
5c. Find the area of R. *[3 marks]*

5d. The region R is now rotated about the y-axis, through 2π radians, to form a solid. *[6 marks]*

By writing $\sin^3 y$ as $\left(1 - \cos^2 y\right) \sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$.

All lengths in this question are in metres.

Consider the function $f(x) = \sqrt{\frac{4-x^2}{8}}$, for $-2 \le x \le 2$. In the following diagram, the shaded region is enclosed by the graph of f and the x-axis.



A container can be modelled by rotating this region by 360° about the x-axis.

6a. Find the volume of the container. *[3 marks]*

Water can flow in and out of the container.

The volume of water in the container is given by the function $g(t)$, for $0 \le t \le 4$, where t is measured in hours and $g(t)$ is measured in m^3. The rate of change of the volume of water in the container is given by $g'(t) = 0.9 - 2.5\cos\left(0.4t^2\right)$.

The volume of water in the container is increasing only when $p < t < q$.

6b. Find the value of p and of q. *[3 marks]*

6c. During the interval $p < t < q$, he volume of water in the container increases by k m^3. Find the value of k. *[3 marks]*

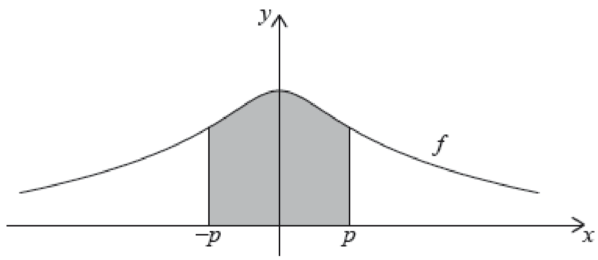
6d. When $t = 0$, the volume of water in the container is 2.3 m^3. It is known that the container is never completely full of water during the 4 hour period. *[5 marks]*

Find the minimum volume of empty space in the container during the 4 hour period.

Let $f(x) = 6 - \ln(x^2 + 2)$, for $x \in \mathbb{R}$. The graph of f passes through the point $(p, 4)$, where $p > 0$.

7a. Find the value of p. *[2 marks]*

7b. The following diagram shows part of the graph of f. *[3 marks]*



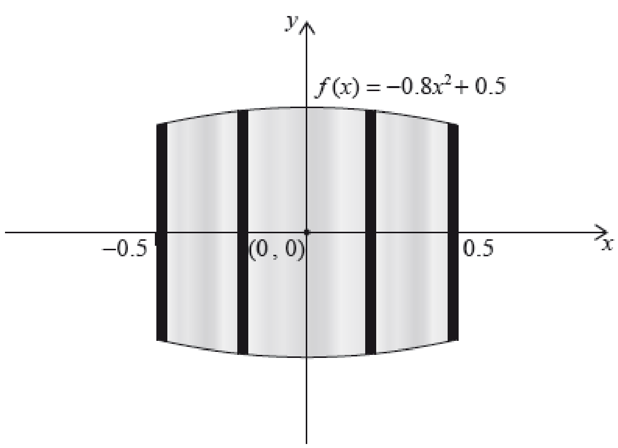
The region enclosed by the graph of f, the x-axis and the lines $x = -p$ and $x = p$ is rotated 360° about the x-axis. Find the volume of the solid formed.

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

8. Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geqslant 0$ and the y-axis for $y \geqslant 0$ is rotated through 2π about the x-axis. *[3 marks]*

All lengths in this question are in metres.

Let $f(x) = -0.8x^2 + 0.5$, for $-0.5 \leqslant x \leqslant 0.5$. Mark uses $f(x)$ as a model to create a barrel. The region enclosed by the graph of f, the x-axis, the line $x = -0.5$ and the line $x = 0.5$ is rotated 360° about the x-axis. This is shown in the following diagram.



9a. Use the model to find the volume of the barrel. *[3 marks]*

9b. The empty barrel is being filled with water. The volume $V\,\mathrm{m}^3$ of water in the barrel after t minutes is given by $V = 0.8(1 - \mathrm{e}^{-0.1t})$. How long will it take for the barrel to be half-full? *[3 marks]*