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Mathematics: analysis and approaches

Practice HL paper 3 questions

For first examinations in 2021

Compactness

[31 marks]

This question asks you to investigate compactness, a quantity that measures how compact an enclosed region is.

The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where A is the area of the region and d is the maximum distance between any two points in the region.

(a) Verify that $C = 1$ for a circular region. [2]

Consider a rectangle whose side lengths are in the ratio $x : 1$.

(b) Show that $C = \frac{4x}{\pi\left(1 + x^2\right)}$. [3]

(c) Show that $\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{4\left(1 - x^2\right)}{\pi\left(1 + x^2\right)^2}$. [2]

(d) (i) Hence find the value of x which gives maximum compactness for a rectangle.

(ii) Identify the geometrical significance of this result. [3]

Consider an equilateral triangle of side length x units.

(e) Find an expression, in terms of x, for the area of this equilateral triangle. [2]

(f) Hence find the exact value of C for an equilateral triangle. [3]

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of radius r units.

(g) If $n > 2$ and even, show that $C = \frac{n}{2\pi} \sin \frac{2\pi}{n}$. [4]

(h) Illustrate, with the aid of a diagram, that $d \neq 2r$ when n is odd. [1]

If $n > 1$ and odd, it can be shown that $C = \frac{n \sin \frac{2\pi}{n}}{\pi\left(1 + \cos \frac{\pi}{n}\right)}$.

(i) Find the regular polygon with the least number of sides for which the compactness is more than 0.995.

[5]

(j) (i) For n even, use L'Hopital's rule or otherwise to show that $\lim_{n \to \infty} C = 1$.

(ii) Using your answer to part (i) show that $\lim_{n \to \infty} C = 1$, for $n > 1$ and odd.

(iii) Interpret geometrically the significance of parts (i) and (ii). [6]

Compactness markscheme:

(a) $A = \pi r^2$ **(A1)**

substituting correctly $A = \pi r^2$ into $C = \frac{4A}{\pi d^2}$ gives $C = \frac{4\pi r^2}{\pi(2r)^2}$ **M1**

leading to $C = 1$ **AG**

[2 marks]

(b) $A = x$ and $d = \sqrt{1+x^2}\left(\Rightarrow d^2 = 1+x^2\right)$ ***(A1)***

substituting correctly $A = x$ and $d = \sqrt{1+x^2}\left(\Rightarrow d^2 = 1+x^2\right)$ into $C = \frac{4A}{\pi d^2}$ ***M1***

$C = \frac{4x}{\pi(1+x^2)}$ ***AG***

[3 marks]

(c) Attempting to use the quotient (or product) rule ***M1***

$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{4\left(\left(1+x^2\right) - x(2x)\right)}{\pi\left(1+x^2\right)^2}$ (or equivalent) ***A1***

$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{4\left(1-x^2\right)}{\pi\left(1+x^2\right)^2}$ ***AG***

[2 marks]

(d) (i) $\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{4\left(1-x^2\right)}{\pi\left(1+x^2\right)^2} = 0 \Rightarrow \left(1-x^2\right) = 0$

(M1)

$\Rightarrow x = 1 \; \left(x > 0\right)$ ***A1***

Note: Do not accept $x = \pm 1$ as a final answer.

(ii) The most compact rectangle is a square ***A1***

[3 marks]

(e) $A = \frac{1}{2}x^2 \sin\frac{\pi}{3}\left(= \frac{\sqrt{3}x^2}{4}\right)$

M1A1

[2 marks]

(f) $d = x$ (for an equilateral triangle) *(A1)*

substituting correctly $A = \frac{1}{2}x^2 \sin\frac{\pi}{3}\left(= \frac{\sqrt{3}x^2}{4}\right)$ and $d = x$ into $C = \frac{4A}{\pi d^2}$ **M1**

$$C = \frac{4\frac{\sqrt{3x^2}}{4}}{\pi x^2}$$

$$= \frac{\sqrt{3}}{\pi}$$

A1

Note: Award ***(A0)(M0)A1*** if $C = \frac{\sqrt{3}}{\pi}$ is obtained by substituting $n = 3$ into $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

[3 marks]

(g) Attempt to use $A = \frac{1}{2}ab\sin C$ with $a = b = r$ and $\theta = \frac{2\pi}{n}$ *(M1)*

each triangle has area $\frac{1}{2}r^2 \sin\frac{2\pi}{n}$ **A1**

(there are n triangles so) $A = \frac{1}{2}nr^2 \sin\frac{2\pi}{n}$ **A1**

$$C = \frac{4\left(\frac{1}{2}nr^2\sin\frac{2\pi}{n}\right)}{\pi(2r)^2}$$ **A1**

leading to $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$ **AG**

[4 marks]

(h) Appropriate diagram showing that the length d is not a diameter when n is odd **A1**

[1 mark]

(i) Attempting to find the least value of n such that $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.995$ *(M1)*

$n = 37$ **A1**

attempting to find the least value of n such that $\dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.995$ ***(M1)***

$n = 29$ ***A1***

Hence a regular 29-sided polygon has the least number of sides such that $C > 0.995$ ***A1***

Note: Award ***(M0)A0(M1)A1*** if $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.995$ is not considered and $\dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.995$ is correctly considered. Award ***(M1)A1(M0)A0*** for $n = 37$ as a final answer.

[5 marks]

(j) (i) Consider $\lim_{n\to\infty}\frac{n}{2\pi}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\left(\dfrac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$ ***M1***

EITHER

$$= \lim_{n\to\infty}\left(\dfrac{-\frac{2\pi}{n^2}\cos\frac{2\pi}{n}}{-\frac{2\pi}{n^2}}\right)$$ ***M1A1***

$$= \lim_{n\to\infty}\left(\cos\frac{2\pi}{n}\right)$$

OR

$\theta = \frac{2\pi}{n}$, as $n \to \infty, \theta \to 0$ ***R1***

$$\lim_{n\to\infty}\left(\dfrac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$ ***A1***

THEN

$= 1$ ***AG***

Note: Allow use of Maclaurin series.

(ii) Consider $\lim_{n\to\infty} C = \lim_{n\to\infty} \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$

$$= \lim_{n\to\infty}\left(\frac{n\sin\frac{2\pi}{n}}{2\pi} \times \frac{2}{\left(1+\cos\frac{\pi}{n}\right)}\right)$$ ***M1***

$$= \left(1 \times \frac{2}{2}\right)$$ ***A1***

$$= 1$$ ***AG***

(iii) As $n \to \infty$, the polygon becomes a circle with compactness = 1 ***A1***

[6 marks]

Total [31 marks]

Gaussian integers

[30 marks]

A **Gaussian integer** is a complex number, z, such that $z = a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\beta = 1 - 2\mathrm{i}$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ.

(a) Find γ. [2]

Now consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\gamma = 11 + 2\mathrm{i}$.

(b) Determine whether $\dfrac{\gamma}{\alpha}$ is a Gaussian integer. [3]

The norm of a complex number z, denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3\mathrm{i}$ then $N(2 + 3\mathrm{i}) = 2^2 + 3^2 = 13$.

(c) On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. [2]

(d) Given that $\alpha = a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. [1]

A **Gaussian prime** is a Gaussian integer, z, that **cannot** be expressed in the form $z = \alpha\beta$ where α, β are Gaussian integers with $N(\alpha), N(\beta) > 1$.

(e) By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. [3]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

(f) Verify that 2 is not a Gaussian prime. [2]

(g) Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. [2]

Let α, β be Gaussian integers.

(h) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. [6]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

(i) Hence show that $1 + 4\mathrm{i}$ is a Gaussian prime. [3]

(j) Use proof by contradiction to prove that a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime. [6]

Gaussian integers markscheme:

(a) $(3+4i)(1-2i)=11-2i$ ***(M1)A1***

[2 marks]

(b) $\frac{\gamma}{\alpha}=\frac{41}{25}-\frac{38}{25}i$ ***(M1)A1***

(Since $\mathrm{Re}\frac{\gamma}{\alpha}\left(=\frac{41}{25}\right)$ and/or $\mathrm{Im}\frac{\gamma}{\alpha}\left(=-\frac{38}{25}\right)$ are not integers)

$\frac{\gamma}{\alpha}$ is not a Gaussian integer ***R1***

Note: Award **R1** for correct conclusion from their answer.

[3 marks]

(c) $\pm 1, \pm i, 0$ plotted and labelled ***A1***

$1\pm i, -1\pm i$ plotted and labelled ***A1***

Note: Award **A1A0** if extra points to the above are plotted and labelled.

[2 marks]

(d) $|z|=\sqrt{a^2+b^2}$ (and as $N(z)=|z|^2$) ***A1***

then $N(z)=a^2+b^2$ ***AG***

[1 mark]

(e) $c^2+d^2=(c+di)(c-di)$ ***A1***

and $N(c+di)=N(c-di)=c^2+d^2$ ***R1***

$N(c+di), N(c-di)>1$ (since c, d are positive) ***R1***

so c^2+d^2 is not a Gaussian prime, by definition ***AG***

[3 marks]

(f) $2\left(=1^2+1^2\right)=(1+i)(1-i)$ ***(A1)***

$N(1+i)=N(1-i)=2$ ***A1***

so 2 is not a Gaussian prime ***AG***

[2 marks]

(g) *For example*, $5\left(=1^2+2^2\right)=(1+2\mathrm{i})(1-2\mathrm{i})$ ***(M1)A1***

[2 marks]

(h) **METHOD 1**

Let $\alpha = m + n\mathrm{i}$ and $\beta = p + q\mathrm{i}$

LHS:

$\alpha\beta = (mp - nq) + (mq + np)\mathrm{i}$ ***M1***

$N(\alpha\beta) = (mp - nq)^2 + (mq + np)^2$ ***A1***

$(mp)^2 - 2mnpq + (nq)^2 + (mq)^2 + 2mnpq + (np)^2$ ***A1***

$(mp)^2 + (nq)^2 + (mq)^2 + (np)^2$ ***A1***

RHS:

$N(\alpha)N(\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$(mp)^2 + (mq)^2 + (np)^2 + (nq)^2$ ***A1***

$\text{LHS} = \text{RHS}$ and so $N(\alpha\beta) = N(\alpha)N(\beta)$ ***AG***

METHOD 2

Let $\alpha = m + n\mathrm{i}$ and $\beta = p + q\mathrm{i}$

LHS

$N(\alpha\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$= (m + n\mathrm{i})(m - n\mathrm{i})(p + q\mathrm{i})(p - q\mathrm{i})$ ***A1***

$= (m + n\mathrm{i})(p + q\mathrm{i})(m - n\mathrm{i})(p - q\mathrm{i})$

$= \left((mp - nq) + (mq + np)\mathrm{i}\right)\left((mp - nq) - (mq + np)\mathrm{i}\right)$ ***M1A1***

$= (mp - nq)^2 + (mq + np)^2$ ***A1***

$= N\left((mp - nq) + (mq + np)\mathrm{i}\right)$ ***A1***

$= N(\alpha)N(\beta)\ (= \text{RHS})$ ***AG***

[6 marks]

(i) $N(1+4\mathrm{i})=17$ which is a prime (in $\mathbb{Z}$) ***R1***

if $1+4\mathrm{i}=\alpha\beta$ then $17=N(\alpha\beta)=N(\alpha)N(\beta)$ ***R1***

we cannot have $N(\alpha),N(\beta)>1$ ***R1***

Note: Award **R1** for stating that $1+4\mathrm{i}$ is not the product of Gaussian integers of smaller norm because no such norms divide 17.

so $1+4\mathrm{i}$ is a Gaussian prime ***AG***

[3 marks]

(j) Assume p is not a Gaussian prime

$\Rightarrow p=\alpha\beta$ where α,β are Gaussian integers and $N(\alpha),N(\beta)>1$ ***M1***

$\Rightarrow N(p)=N(\alpha)N(\beta)$ ***M1***

$p^2=N(\alpha)N(\beta)$ ***A1***

It cannot be $N(\alpha)=1,N(\beta)=p^2$ from definition of Gaussian prime ***R1***

hence $N(\alpha)=p,N(\beta)=p$ ***R1***

If $\alpha=a+b\mathrm{i}$ then $N(\alpha)=a^2+b^2=p$ which is a contradiction **R1**

hence a prime number, p, that is not of the form a^2+b^2 is a Gaussian prime ***AG***

[6 marks]

Total [30 marks]