1. The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

(a) Find the value of p and the value of q.

(4)

(b) The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph.

(2)
(Total 6 marks)

(a) **METHOD 1**

$f'(x) = q - 2x = 0$	M1
$f'(3) = q - 6 = 0$	
$q = 6$	A1
$f(3) = p + 18 - 9 = 5$	M1
$p = -4$	A1

METHOD 2

$f(x) = -(x-3)^2 + 5$	M1A1
$= -x^2 + 6x - 4$	
$q = 6, p = -4$	A1A1

(b) $g(x) = -4 + 6(x-3) - (x-3)^2\ (= -31 + 12x - x^2)$ M1A1

Note: Accept any alternative form that is correct.
Award M1A0 for a substitution of $(x + 3)$.

[6]

2. When the function $q(x) = x^3 + kx^2 - 7x + 3$ is divided by $(x + 1)$ the remainder is seven times the remainder that is found when the function is divided by $(x + 2)$.

Find the value of k.

(Total 5 marks)

$q(-1) = k + 9$	M1A1
$q(-2) = 4k + 9$	A1
$k + 9 = 7(4k + 9)$	M1
$k = -2$	A1

Notes: The first M1 is for one substitution and the consequent equations.
Accept expressions for $q(-1)$ and $q(-2)$ that are not simplified.

[5]

3. Let $g(x) = \log_5 \left| 2\log_3 x \right|$. Find the product of the zeros of g.

(Total 5 marks)

$g(x) = 0$

$\log_5|2\log_3 x| = 0$ (M1)

$|2\log_3 x| = 1$ A1

$\log_3 x = \pm\frac{1}{2}$ (A1)

$x = 3^{\pm\frac{1}{2}}$ A1

so the product of the zeros of g is $3^{\frac{1}{2}} \times 3^{-\frac{1}{2}} = 1$ A1 N0

[5]

4. The sum, S_n, of the first n terms of a geometric sequence, whose n^{th} term is u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n}, \text{ where } a > 0.$$

(a) Find an expression for u_n.

(2)

(b) Find the first term and common ratio of the sequence.

(4)

(c) Consider the sum to infinity of the sequence.

(i) Determine the values of a such that the sum to infinity exists.

(ii) Find the sum to infinity when it exists.

(2)
(Total 8 marks)

(a) $u_n = S_n - S_{n-1}$ (M1)

$= \dfrac{7^n - a^n}{7^n} - \dfrac{7^{n-1} - a^{n-1}}{7^{n-1}}$ A1

(b) **EITHER**

$u_1 = 1 - \dfrac{a}{7}$ A1

$u_2 = 1 - \dfrac{a^2}{7^2} - \left(1 - \dfrac{a}{7}\right)$ M1

$= \dfrac{a}{7}\left(1 - \dfrac{a}{7}\right)$ A1

common ratio $= \dfrac{a}{7}$ A1

OR

$u_n = 1 - \left(\dfrac{a}{7}\right)^n - 1 + \left(\dfrac{a}{7}\right)^{n-1}$ M1

$= \left(\dfrac{a}{7}\right)^{n-1}\left(1 - \dfrac{a}{7}\right)$

$= \dfrac{7-a}{7}\left(\dfrac{a}{7}\right)^{n-1}$ A1

$u_1 = \dfrac{7-a}{7}$, common ratio $= \dfrac{a}{7}$ A1A1

(c) (i) $0 < a < 7$ (accept $a < 7$) A1

(ii) 1 A1

5. (a) Express the quadratic $3x^2 - 6x + 5$ in the form $a(x + b)^2 + c$, where $a, b, c \in \mathbb{Z}$.

(3)

(b) Describe a sequence of transformations that transforms the graph of $y = x^2$ to the graph of $y = 3x^2 - 6x + 5$.

(3)

(Total 6 marks)

(a) attempt at completing the square (M1)

$3x^2 - 6x + 5 = 3(x^2 - 2x) + 5 = 3\,(x - 1)^2 - 1 + 5$ (A1)

$= 3(x - 1)^2 + 2$ A1

$(a = 3, b = -1, c = 2)$

(b) definition of suitable basic transformations:

T_1 = stretch in y direction scale factor 3 A1

T_2 = translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ A1

T_3 = translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ A1

[6]

6. Given that $\dfrac{z}{z+2} = 2 - \text{i}$, $z \in \mathbb{C}$, find z in the form $a + \text{i}b$.

(Total 4 marks)

$z = (2 - \text{i})(z + 2)$ M1

$\quad = 2z + 4 - \text{i}z - 2\text{i}$

$z(1 - \text{i}) = -4 + 2\text{i}$

$z = \dfrac{-4 + 2\text{i}}{1 - \text{i}}$ A1

$z = \dfrac{-4 + 2\text{i}}{1 - \text{i}} \times \dfrac{1 + \text{i}}{1 + \text{i}}$ M1

$= -3 - \text{i}$ A1

7. A geometric sequence $u_1, u_2, u_3, \ldots$ has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

(a) Find the common ratio of the geometric sequence.

(2)

An arithmetic sequence $v_1, v_2, v_3, \ldots$ is such that $v_2 = u_2$ and $v_4 = u_4$.

(b) Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$.

(5)
(Total 7 marks)

(a) $u_1 = 27$

$\frac{81}{2} = \frac{27}{1-r}$ M1

$r = \frac{1}{3}$ A1

(b) $v_2 = 9$

$v_4 = 1$

$2d = -8 \Rightarrow d = -4$ (A1)

$v_1 = 13$ (A1)

$\frac{N}{2}(2 \times 13 - 4(N-1)) > 0$ (accept equality) M1

$\frac{N}{2}(30 - 4N) > 0$

$N(15 - 2N) > 0$

$N < 7.5$ (M1)

$N = 7$ A1

Note: $13 + 9 + 5 + 1 - 3 - 7 - 11 > 0 \Rightarrow N = 7$ or equivalent receives full marks.

[7]

8. (a) Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

(2)

(b) Hence find the value of $\cot\frac{\pi}{8}$ in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

(3)
(Total 5 marks)

(a) $\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta - 1}$ M1

Note: Award M1 for use of double angle formulae.

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$ A1

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$ AG

(b) $\tan\frac{\pi}{8} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ (M1)

$\cot\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$ M1

$= \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= 1 + \sqrt{2}$ A1

[5]

9. Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$.

(Total 6 marks)

area = $\int_0^{\sqrt{3}} \arctan x \, dx$ A1

attempting to integrate by parts M1

$= [x \arctan x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x \frac{1}{1+x^2} dx$ A1A1

$= [x \arctan x]_0^{\sqrt{3}} - \left[\frac{1}{2} \ln(1+x^2)\right]_0^{\sqrt{3}}$ A1

Note: Award A1 even if limits are absent.

$= \frac{\pi}{\sqrt{3}} - \frac{1}{2} \ln 4$ A1

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

10. Show that the points (0, 0) and $(\sqrt{2\pi}, -\sqrt{2\pi})$ on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

(Total 7 marks)

Attempt at implicit differentiation M1

$e^{(x+y)}\left(1 + \frac{dy}{dx}\right) = -\sin(xy)\left(x\frac{dy}{dx} + y\right)$ A1A1

let $x = 0, y = 0$ M1

$e^0\left(1 + \frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} = -1$ A1

let $x = \sqrt{2\pi}, y = -\sqrt{2\pi}$

$e^0\left(1 + \frac{dy}{dx}\right) = -\sin(-2\pi)\left(x\frac{dy}{dx} + y\right) = 0$

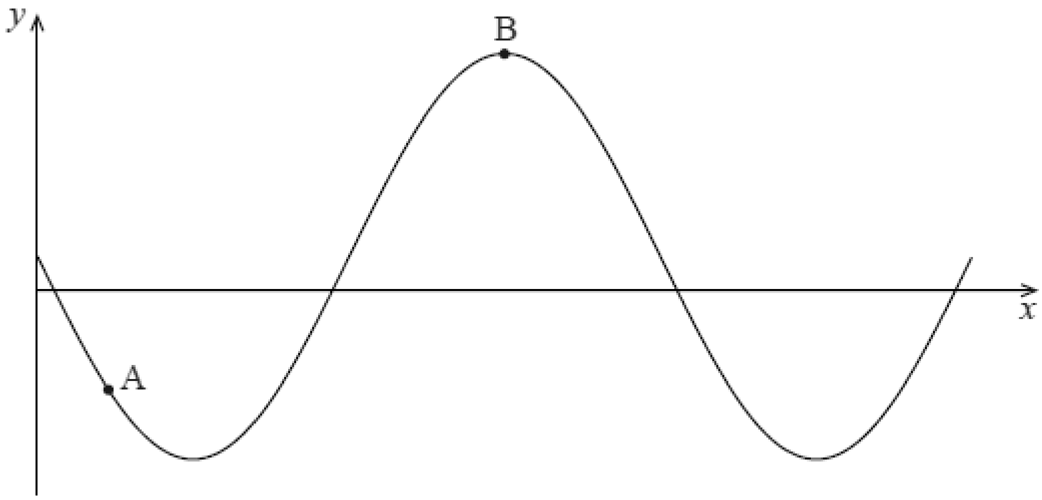
so $\frac{dy}{dx} = -1$ A1

since both points lie on the line $y = -x$ this is a common tangent R1

Note: $y = -x$ must be seen for the final R1. It is not sufficient to note that the gradients are equal.

[7]

11. The diagram below shows a curve with equation $y = 1 + k \sin x$, defined for $0 \le x \le 3\pi$.



The point $A\left(\frac{\pi}{6}, -2\right)$ lies on the curve and $B(a, b)$ is the maximum point.

(a) Show that $k = -6$.

(2)

(b) Hence, find the values of a and b.

(3)
(Total 5 marks)

(a) $-2 = 1 + k\sin\left(\frac{\pi}{6}\right)$ M1

$-3 = \frac{1}{2}k$ A1

$k = -6$ AG N0

(b) **METHOD 1**

maximum $\Rightarrow \sin x = -1$ M1

$a = \frac{3\pi}{2}$ A1

$b = 1 - 6(-1)$
$= 7$ A1 N2

METHOD 2

$y' = 0$ M1

$k \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$a = \frac{3\pi}{2}$ A1

$b = 1 - 6(-1)$
$= 7$ A1 N2

Note: Award A1A1 for $(\frac{3\pi}{2}, 7)$

[5]

12. Consider the function $f : x \to \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of *f*.

(4)

(b) Determine an expression for the inverse function, f^{-1}, and write down its domain.

(4)
(Total 8 marks)

(a) $\frac{\pi}{4} - \arccos x \geq 0$

$\arccos x \leq \frac{\pi}{4}$ (M1)

$x \geq \frac{\sqrt{2}}{2}$ $\left(\text{accept } x \geq \frac{1}{\sqrt{2}}\right)$ (A1)

since $-1 \leq x \leq 1$ (M1)

$\Rightarrow \frac{\sqrt{2}}{2} \leq x \leq 1$ $\left(\text{accept } \frac{1}{\sqrt{2}} \leq x \leq 1\right)$ A1

Note: Penalize the use of $<$ instead of $\leq$ only once.

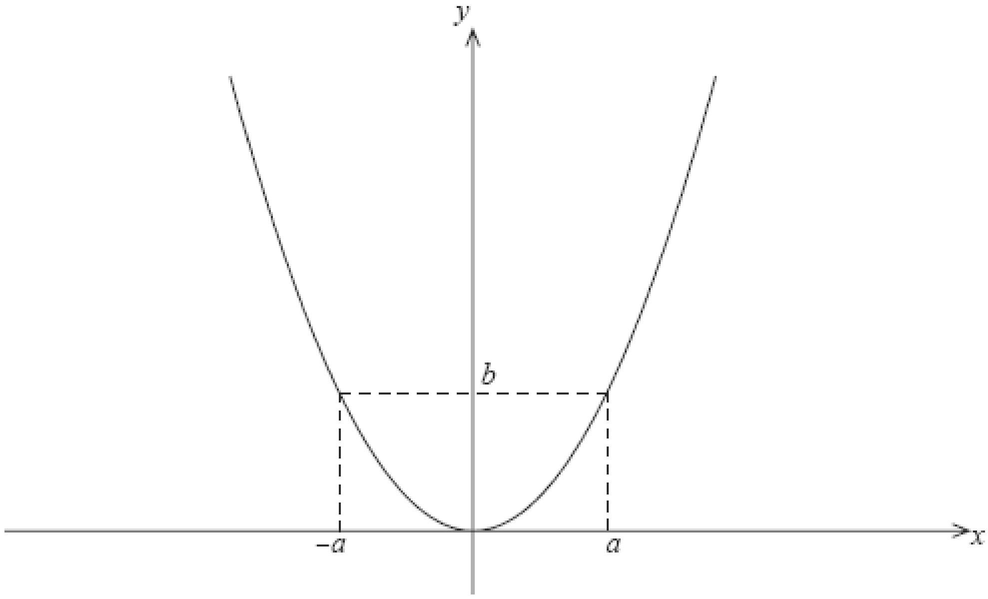
(b) $y = \sqrt{\frac{\pi}{4} - \arccos x} \Rightarrow x = \cos\left(\frac{\pi}{4} - y^2\right)$ M1A1

$f^{-1} : x \to \cos\left(\frac{\pi}{4} - x^2\right)$ A1

$0 \leq x \leq \sqrt{\frac{\pi}{4}}$ A1

[8]

13. The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$, where $b > a > 0$.

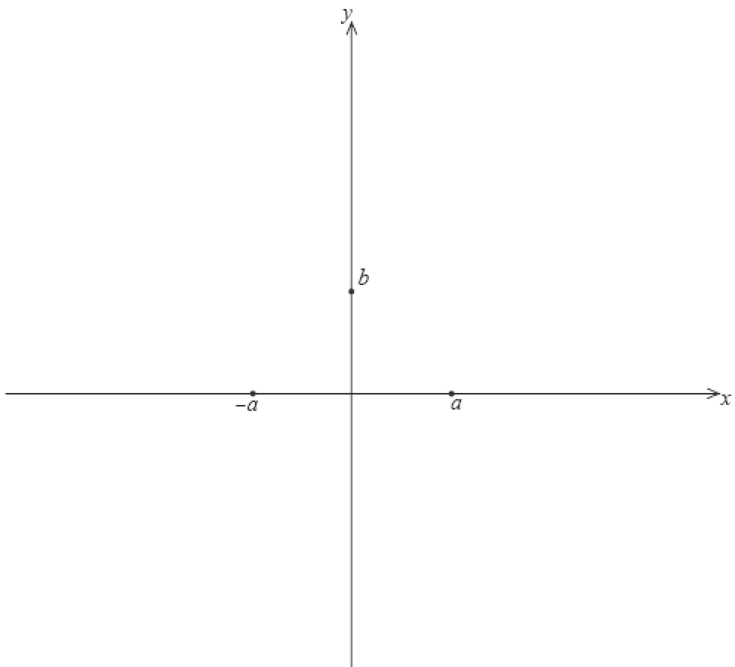


Consider the function $g(x) = \dfrac{1}{f(x-a)-b}$.

(a) Find the largest possible domain of the function g.

(2)

(b) On the axes below, sketch the graph of $y = g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates.



(6)
(Total 8 marks)

(a) $f(x-a) \neq b$ (M1)
$x \neq 0$ and $x \neq 2a$ (or equivalent) A1

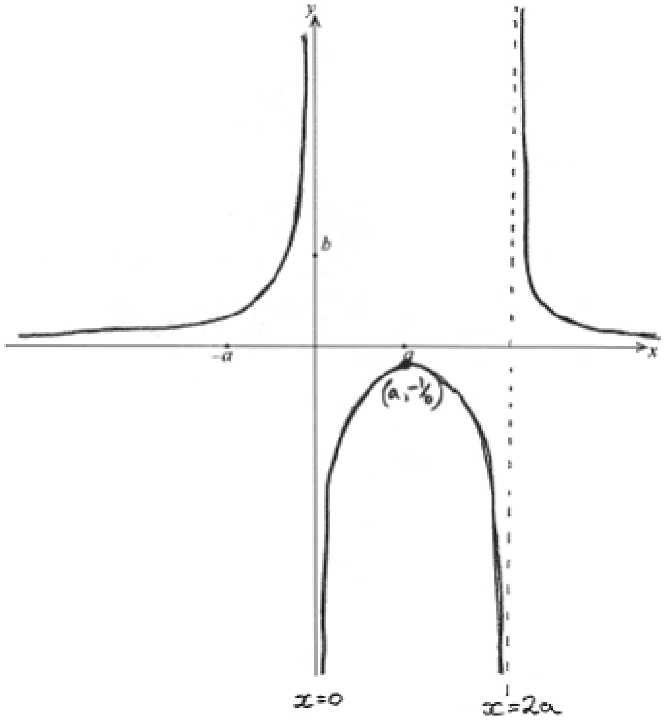
(b) vertical asymptotes $x = 0$, $x = 2a$ A1
horizontal asymptote $y = 0$ A1

Note: Equations must be seen to award these marks.

maximum $\left(a, -\frac{1}{b}\right)$ A1A1

Note: Award A1 for correct x-coordinate and A1 for correct y-coordinate.

one branch correct shape A1
other 2 branches correct shape A1



[8]

14. The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \text{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\text{A}\hat{\text{O}}\text{B}$ in terms of π.

(3)
(Total 6 marks)

(a) $\text{AB} = \sqrt{1^2 + (2 - \sqrt{3})^2}$ M1

$= \sqrt{88 - 4\sqrt{3}}$ A1

$= 2\sqrt{2 - \sqrt{3}}$ A1

(b) **METHOD 1**

$\arg z_1 = -\frac{\pi}{4}, \ \arg z_2 = -\frac{\pi}{3}$ A1A1

Note: Allow $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Note: Allow degrees at this stage.

$\text{A}\hat{\text{O}}\text{B} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ (accept $-\frac{\pi}{12}$) A1

Note: Allow FT for final A1.

METHOD 2

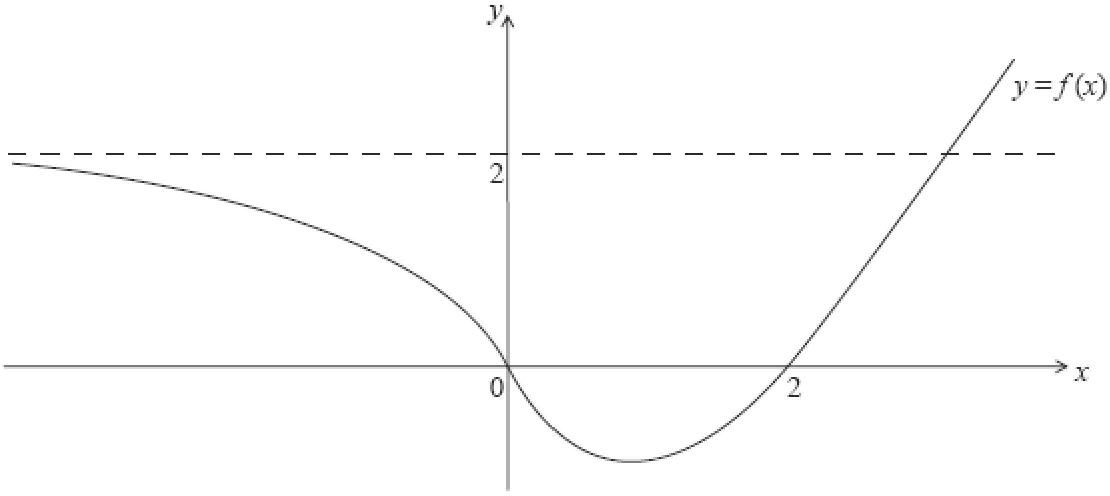
attempt to use scalar product or cosine rule M1

$\cos \text{A}\hat{\text{O}}\text{B} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$ A1

$\text{A}\hat{\text{O}}\text{B} = \frac{\pi}{12}$ A1

[6]

15. The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.



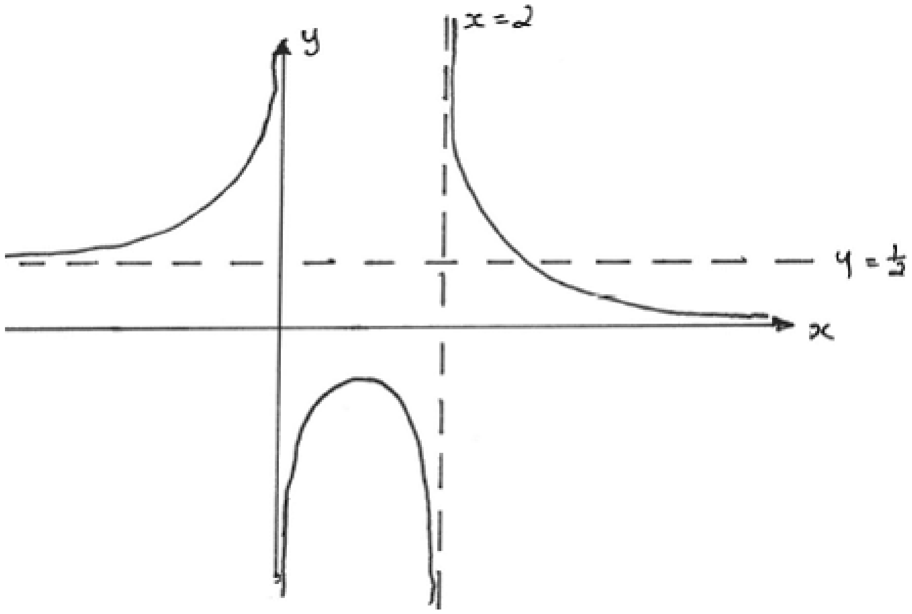
(a) Sketch the graph of $y = \dfrac{1}{f(x)}$.

(3)

(b) Sketch the graph of $y = x\,f(x)$.

(3)
(Total 6 marks)

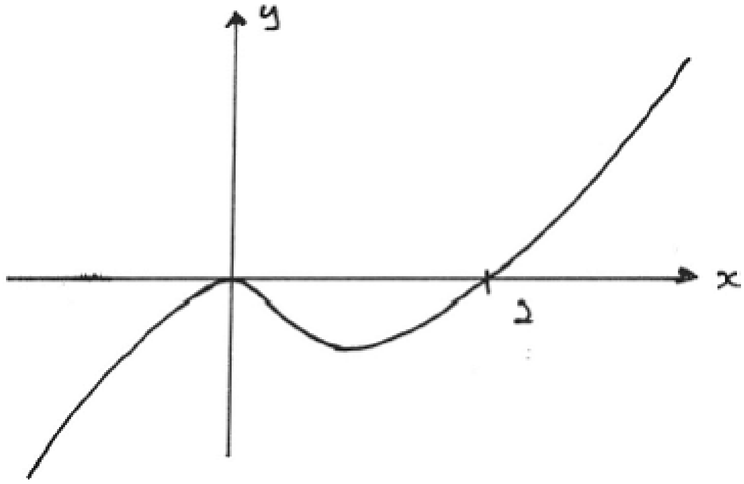
(a)



A3

Note: Award A1 for each correct branch with position of asymptotes clearly indicated. If $x = 2$ is not indicated, only penalise once.

(b)



A3

Note: Award A1 for behaviour at $x = 0$, A1 for intercept at $x = 2$, A1 for behaviour for large $|x|$.

[6]

16. Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$.

(Total 7 marks)

$2 + x - x^2 = 2 - 3x + x^2$ M1

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x - 2) = 0$

$\Rightarrow x = 0, x = 2$ A1A1

Notes: Accept graphical solution.
Award M1 for correct graph and A1A1 for correctly labelled roots.

$\therefore A = \int_0^2 \left(\left(2 + x - x^2\right) - \left(2 - 3x + x^2\right)\right) dx$ (M1)

$= \int_0^2 \left(4x - 2x^2\right) dx$ or equivalent A1

$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$ A1

$= \frac{8}{3}\left(= 2\frac{2}{3}\right)$ A1

[7]

17. In triangle ABC, AB = 9 cm, AC =12 cm, and $\hat{B}$ is twice the size of $\hat{C}$.

Find the cosine of $\hat{C}$.

(Total 5 marks)

$\frac{9}{\sin C} = \frac{12}{\sin B}$ (M1)

$\frac{9}{\sin C} = \frac{12}{\sin 2C}$ A1

Using double angle formula $\frac{9}{\sin C} = \frac{12}{2 \sin C \cos C}$ M1

$\Rightarrow 9(2 \sin C \cos C) = 12 \sin C$

$\Rightarrow 6 \sin C (3 \cos C - 2) = 0$ or equivalent (A1)

$(\sin C \neq 0)$

$\Rightarrow \cos C = \frac{2}{3}$ A1

[5]

18. Given that $z_1 = 2$ and $z_2 = 1 + \mathrm{i}\sqrt{3}$ are roots of the cubic equation $z^3 + bz^2 + cz + d = 0$ where $b, c, d \in \mathbb{R}$,

(a) write down the third root, z_3, of the equation;

(1)

(b) find the values of b, c and d;

(4)
(Total 5 marks)

(a) $1 - \mathrm{i}\sqrt{3}$ A1

(b) **EITHER**

$(z - (1 + \mathrm{i}\sqrt{3}))(z - (1 - \mathrm{i}\sqrt{3})) = z^2 - 2z + 4$ (M1)A1

$p(z) = (z - 2)(z^2 - 2z + 4)$ (M1)

$= z^3 - 4z^2 + 8z - 8$ A1

therefore $b = -4$, $c = 8$, $d = -8$

OR

relating coefficients of cubic equations to roots

$-b = 2 + 1 + \mathrm{i}\sqrt{3} + 1 - \mathrm{i}\sqrt{3} = 4$ M1

$c = 2\,(1 + \mathrm{i}\sqrt{3}) + 2\,(1 - \mathrm{i}\sqrt{3}) + (1 + \mathrm{i}\sqrt{3})(1 - \mathrm{i}\sqrt{3}) = 8$

$-d = 2(1 + \mathrm{i}\sqrt{3})(1 - \mathrm{i}\sqrt{3}) = 8$

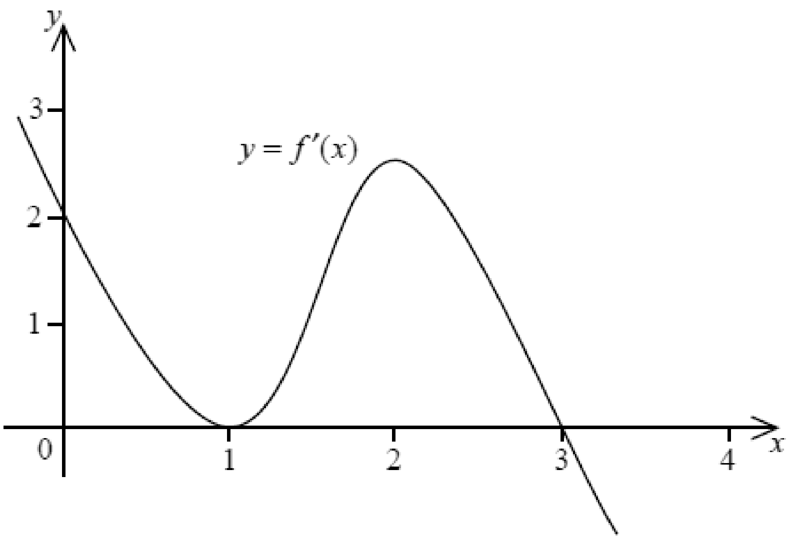
$b = -4$, $c = 8$, $d = -8$ A1A1A1

(c) $z_2 = 2\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}$, $z_3 = 2\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}$ A1A1A1

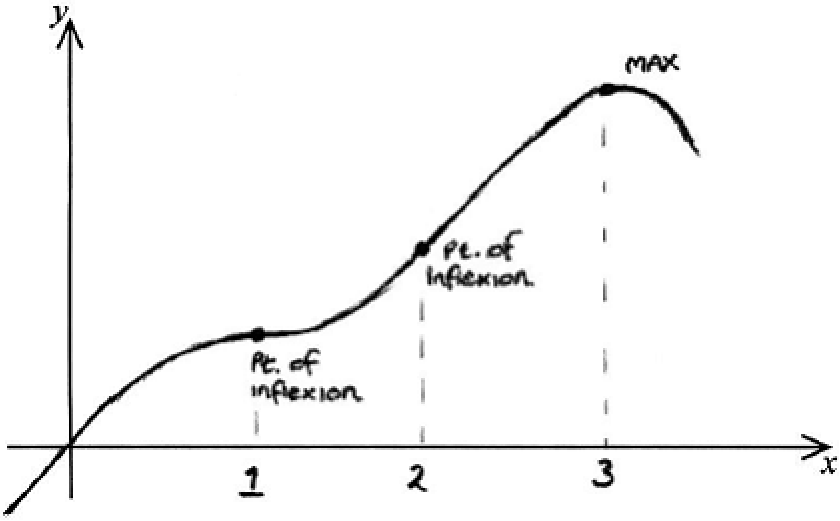
Note: Award A1 for modulus,
A1 for each argument.

[8]

19. The diagram below shows a sketch of the gradient function $f'(x)$ of the curve $f(x)$.



On the graph below, sketch the curve $y = f(x)$ given that $f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.



A5

Note: Award A1 for origin
A1 for shape
A1 for maximum
A1 for each point of inflexion.

[5]

(Total 5 marks)

20. The curve C has equation $y = \frac{1}{8}(9+8x^2-x^4)$.

(a) Find the coordinates of the points on C at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

(4)

(b) The tangent to C at the point P(1, 2) cuts the x-axis at the point T. Determine the coordinates of T.

(4)

(c) The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN.

(7)

(Total 15 marks)

(a) $\frac{dy}{dx} = 2x - \frac{1}{2}x^3$ A1

$x\left(2 - \frac{1}{2}x^2\right) = 0$

$x = 0, \pm 2$

$\frac{dy}{dx} = 0$ at $\left(0, \frac{9}{8}\right), \left(-2, \frac{25}{8}\right), \left(2, \frac{25}{8}\right)$ A1A1A1

Note: Award A2 for all three x-values correct with errors/omissions in y-values.

(b) at $x = 1$, gradient of tangent $= \frac{3}{2}$ (A1)

Note: In the following, allow FT on incorrect gradient.

equation of tangent is $y - 2 = \frac{3}{2}(x - 1)$ $\left(y = \frac{3}{2}x + \frac{1}{2}\right)$ (A1)

meets x-axis when $y = 0$, $-2 = \frac{3}{2}(x - 1)$ (M1)

$x = -\frac{1}{3}$

coordinates of T are $\left(-\frac{1}{3}, 0\right)$ A1

(c) gradient of normal $= -\frac{2}{3}$ (A1)

equation of normal is $y - 2 = -\frac{2}{3}(x - 1)$ $\left(y = -\frac{2}{3}x + \frac{8}{3}\right)$ (M1)

at $x = 0$, $y = \frac{8}{3}$ A1

Note: In the following, allow FT on incorrect coordinates of T and N.

lengths of PN $= \sqrt{\frac{13}{9}}$, PT $= \sqrt{\frac{52}{9}}$ A1A1

area of triangle PTN $= \frac{1}{2} \times \sqrt{\frac{13}{9}} \times \sqrt{\frac{52}{9}}$ M1

$= \frac{13}{9}$ (or equivalent e.g. $\frac{\sqrt{676}}{18}$) A1

[15]

21. The normal to the curve $xe^{-y} + e^{y} = 1 + x$, at the point $(c, \ln c)$, has a y-intercept $c^2 + 1$.

Determine the value of c.

(Total 7 marks)

EITHER

differentiating implicitly:

$1 \times e^{-y} - xe^{-y}\frac{dy}{dx} + e^{y}\frac{dy}{dx} = 1$ M1A1

at the point $(c, \ln c)$

$\frac{1}{c} - c \times \frac{1}{c}\frac{dy}{dx} + c\frac{dy}{dx} = 1$ M1

$\frac{dy}{dx} = \frac{1}{c}$ $(c \neq 1)$ (A1)

OR

reasonable attempt to make expression explicit (M1)

$xe^{-y} + e^{y} = 1 + x$

$x + e^{2y} = e^{y}(1 + x)$

$e^{2y} - e^{y}(1 + x) + x = 0$

$(e^{y} - 1)(e^{y} - x) = 0$ (A1)

$e^{y} = 1,\ e^{y} = x$

$y = 0,\ y = \ln x$ A1

Note: Do not penalize if $y = 0$ not stated.

$\frac{dy}{dx} = \frac{1}{2}$

gradient of tangent $= \frac{1}{c}$ A1

Note: If candidate starts with $y = \ln x$ with no justification, award (M0)(A0)A1A1.

THEN

the equation of the normal is

$y - \ln c = -c(x - c)$ M1

$x = 0,\ y = c^2 + 1$

$c^2 + 1 - \ln c = c^2$ (A1)

$\ln c = 1$

$c = e$ A1

[7]

22. (a) (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leq x \leq \frac{\pi}{2}$.

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leq x \leq \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

(9)

(b) Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$.

(8

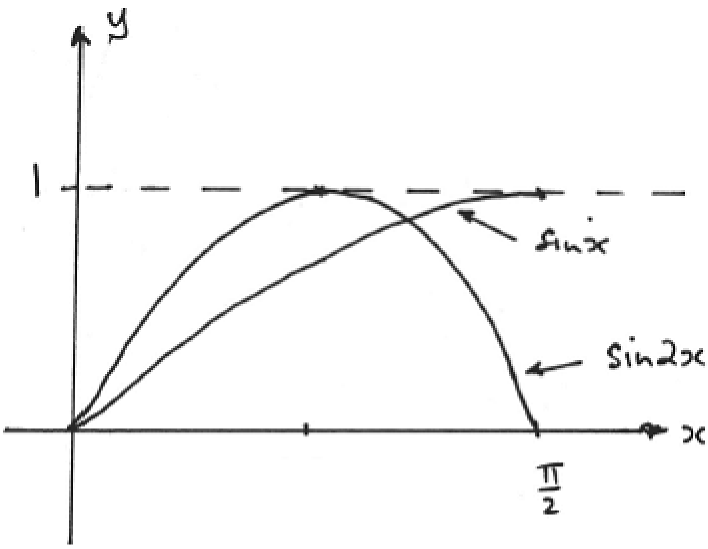
(c) The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$.

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

(8)
(Total 25 marks)

(a) (i)



A2

Note: Award A1 for correct $\sin x$, A1 for correct $\sin 2x$.

Note: Award A1A0 for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x,\ 0 \le x \le \frac{\pi}{2}$

$2 \sin x \cos x - \sin x = 0$ M1

$\sin x\,(2 \cos x - 1) = 0$

$x = 0,\ \frac{\pi}{3}$ A1A1 N1N1

(iii) area $= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx$ M1

Note: Award M1 for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ A1

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ (M1)

$= \frac{3}{4} - \frac{1}{2}$

$= \frac{1}{4}$ A1

(b) $\int_0^1 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\,\mathrm{d}\theta$ M1A1A1

Note: Award M1 for substitution and reasonable attempt at finding expression for $\mathrm{d}x$ in terms of $\mathrm{d}\theta$, first A1 for correct limits, second A1 for correct substitution for $\mathrm{d}x$.

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\,\mathrm{d}\theta$ A1

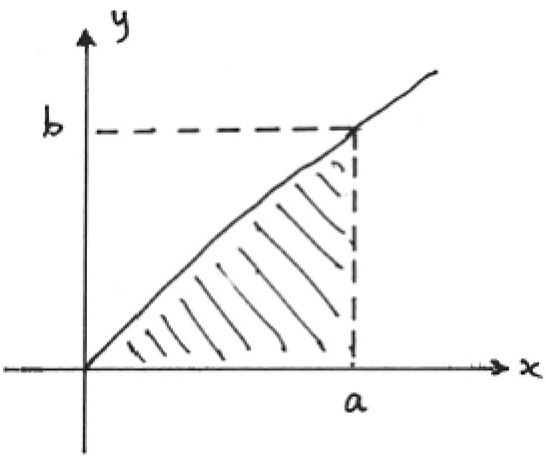
$\int_0^{\frac{\pi}{6}} 4-4\cos 2\theta\,\mathrm{d}\theta$ M1

$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ A1

$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ (M1)

$= \frac{2\pi}{3} - \sqrt{3}$ A1

(c) (i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$ R1

$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$ AG

(ii) $f(x) = \arcsin\frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ A1

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\mathrm{d}x$ M1A1A1

Note: Award A1 for the limit $\frac{\pi}{6}$ seen anywhere, A1 for all else correct.

$= \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$ A1

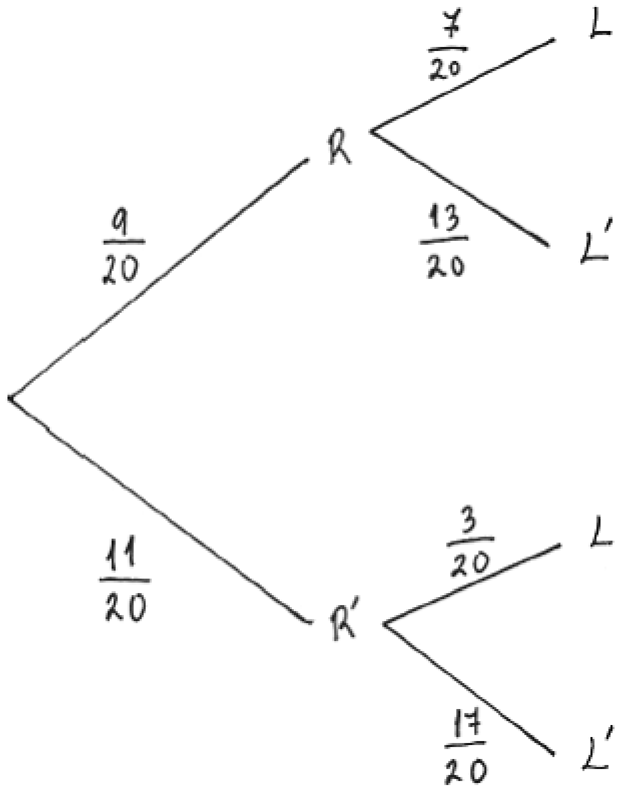
$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ A1

Note: Award no marks for methods using integration by parts.

[25]

23. Jenny goes to school by bus every day. When it is not raining, the probability that the bus is late is $\frac{3}{20}$. When it is raining, the probability that the bus is late is $\frac{7}{20}$. The probability that it rains on a particular day is $\frac{9}{20}$. On one particular day the bus is late. Find the probability that it is not raining on that day.

(Total 5 marks)



(A1)

$\mathrm{P}(R' \cap L) = \frac{11}{20} \times \frac{3}{20}$ A1

$\mathrm{P}(L) = \frac{9}{20} \times \frac{7}{20} + \frac{11}{20} \times \frac{3}{20}$ A1

$P(R'|L) = \frac{\mathrm{P}(R' \cap L)}{\mathrm{P}(L)}$ (M1)

$= \frac{33}{96} \left(= \frac{11}{32} \right)$ A1

[5]

24. Given that $Ax^3 + Bx^2 + x + 6$ is exactly divisible by $(x + 1)(x - 2)$, find the value of A and the value of B.

(Total 5 marks)

using the factor theorem or long division	M1	
$-A + B - 1 + 6 = 0 \Rightarrow A - B = 5$	A1	
$8A + 4B + 2 + 6 = 0 \Rightarrow 2A + B = -2$	A1	
$3A = 3 \Rightarrow A = 1$	A1	
$B = -4$	A1	N3

Note: Award M1A0A0A1A1 for using $(x - 3)$ as the third factor, without justification that the leading coefficient is 1.

[5]

25. (a) Show that $\dfrac{3}{x+1} + \dfrac{2}{x+3} = \dfrac{5x+11}{x^2+4x+3}$.

(2)

(b) Hence find the value of k such that $\displaystyle\int_0^2 \frac{5x+11}{x^2+4x+3}\,dx = \ln k$.

(4)

(Total 6 marks)

(a) $\dfrac{3}{x+1} + \dfrac{2}{x+3} = \dfrac{3(x+3)+2(x+1)}{(x+1)(x+3)}$ M1

$= \dfrac{3x+9+2x+2}{x^2+4x+3}$ A1

$= \dfrac{5x+11}{x^2+4x+3}$ AG

(b) $\displaystyle\int_0^2 \frac{5x+11}{x^2+4x+3}dx = \int_0^2 \left(\frac{3}{x+1} + \frac{2}{x+3}\right)dx$ M1

$= [3\ln(x+1) + 2\ln(x+3)]_0^2$ A1

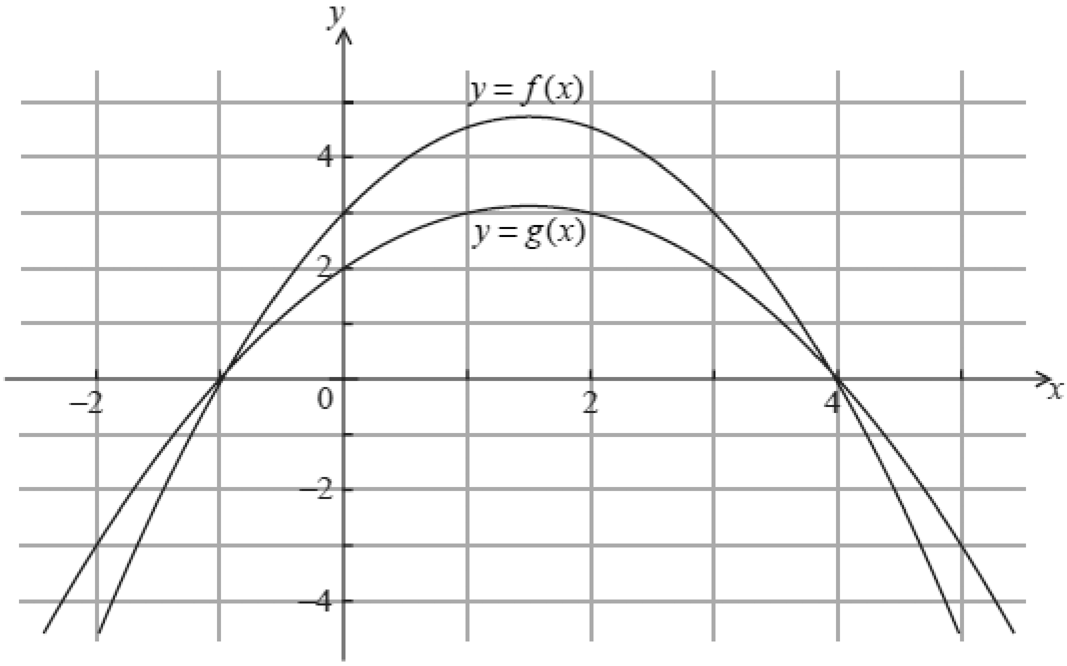
$= 3\ln 3 + 2\ln 5 - 3\ln 1 - 2\ln 3 \quad (= 3\ln 3 + 2\ln 5 - 2\ln 3)$ A1

$= \ln 3 + 2\ln 5$

$= \ln 75 \ (k = 75)$ A1

[6]

26. Shown below are the graphs of $y = f(x)$ and $y = g(x)$.



If $(f \circ g)(x) = 3$, find all possible values of x.

(Total 4 marks)

$g(x) = 0$ or 3 (M1)(A1)
$x = -1$ or 4 or 1 or 2 A1A1

Notes: Award A1A1 for all four correct values,
A1A0 for two or three correct values,
A0A0 for less than two correct values.

Award M1 and corresponding *A* marks for correct attempt to find expressions for *f* and *g*.

[4]

27. (a) Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x$.

(6)

(b) Find $\int \tan^3 x \, \mathrm{d}x$.

(3)
(Total 9 marks)

(a) **EITHER**

let $u = \tan x$; $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$ (M1)
consideration of change of limits (M1)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u^{\frac{1}{3}}} \mathrm{d}u \qquad \text{(A1)}$$

Note: Do not penalize lack of limits.

$$= \left[\frac{3u^{\frac{2}{3}}}{2} \right]_1^{\sqrt{3}} \qquad \text{A1}$$

$$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2} \right) \qquad \text{A1A1} \quad \text{N0}$$

OR

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \left[\frac{3(\tan x)^{\frac{2}{3}}}{2} \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \qquad \text{M2A2}$$

$$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2} \right) \qquad \text{A1A1} \quad \text{N0}$$

(b) $\int \tan^3 x \mathrm{d}x = \int \tan x (\sec^2 x - 1) \mathrm{d}x$ M1

$= \int (\tan x \times \sec^2 x - \tan x) \mathrm{d}x$

$= \frac{1}{2} \tan^2 x - \ln|\sec x| + C$ A1A1

Note: Do not penalize the absence of absolute value or C.

[9]

28. The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius R and the inner circle has radius r.

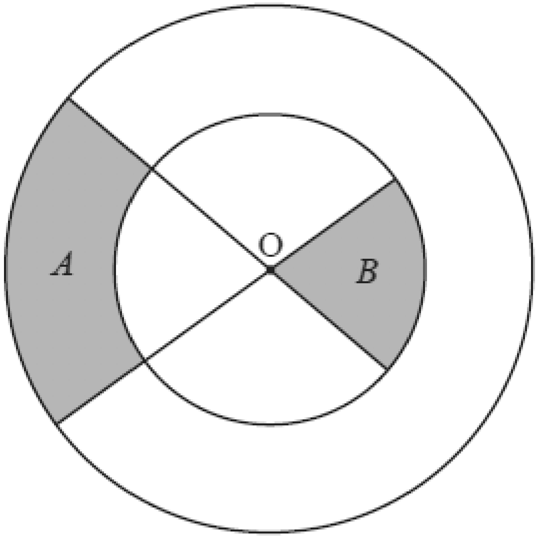


diagram not to scale

Consider the shaded regions with areas A and B. Given that $A : B = 2 : 1$, find the **exact** value of the ratio $R : r$.

(Total 5 marks)

$A = \frac{\theta}{2}(R^2 - r^2)$ A1

$B = \frac{\theta}{2}r^2$ A1

from $A : B = 2:1$, we have $R^2 - r^2 = 2r^2$ M1

$R = \sqrt{3}r$ (A1)

hence exact value of the ratio $R : r$ is $\sqrt{3} : 1$ A1 N0

[5]

29. If x satisfies the equation $\sin\left(x+\frac{\pi}{3}\right)=2\sin x\sin\left(\frac{\pi}{3}\right)$, show that 11 tan $x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

(Total 6 marks)

$\sin\left(x+\frac{\pi}{3}\right)=\sin x\cos\left(\frac{\pi}{3}\right)+\cos x\sin\left(\frac{\pi}{3}\right)$	(M1)
$\sin x\cos\left(\frac{\pi}{3}\right)+\cos x\sin\left(\frac{\pi}{3}\right)=2\sin x\sin\left(\frac{\pi}{3}\right)$	
$\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x=2\times\frac{\sqrt{3}}{2}\sin x$	A1
dividing by $\cos x$ and rearranging	M1
$\tan x=\frac{\sqrt{3}}{2\sqrt{3}-1}$	A1
rationalizing the denominator	M1
$11\tan x = 6+\sqrt{3}$	A1

[6]

30. The common ratio of the terms in a geometric series is 2^x.

(a) State the set of values of x for which the sum to infinity of the series exists.

(2)

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40.

(4)

(Total 6 marks)

(a)	$0 < 2^x < 1$	(M1)	
	$x < 0$	A1	N2
(b)	$\frac{35}{1-r}=40$	M1	
	$\Rightarrow 40 - 40\times r = 35$		
	$\Rightarrow -40\times r = -5$	(A1)	
	$\Rightarrow r = 2^x = \frac{1}{8}$	A1	
	$\Rightarrow x=\log_2\frac{1}{8}\ (=-3)$	A1	

Note: The substitution $r = 2^x$ may be seen at any stage in the solution.

[6]

31. Consider $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 5x + 4}$.

(a) Find the equations of all asymptotes of the graph of f.

(4)

(b) Find the coordinates of the points where the graph of f meets the x and y axes.

(2)

(c) Find the coordinates of

(i) the maximum point and justify your answer;

(ii) the minimum point and justify your answer.

(10)

(d) Sketch the graph of f, clearly showing all the features found above.

(3)

(e) **Hence**, write down the number of points of inflexion of the graph of f.

(1)
(Total 20 marks)

(a) $x^2 + 5x + 4 = 0 \Rightarrow x = -1$ or $x = -4$ (M1)
so vertical asymptotes are $x = -1$ and $x = -4$ A1
as $x \to \infty$ then $y \to 1$ so horizontal asymptote is $y = 1$ (M1)A1

(b) $x^2 - 5x + 4 = 0 \Rightarrow x = 1$ or $x = 4$ A1
$x = 0 \Rightarrow y = 1$
so intercepts are (1, 0), (4, 0) and (0, 1)

(c) (i) $f'(x) = \dfrac{(x^2 + 5x + 4)(2x - 5) - (x^2 - 5x + 4)(2x + 5)}{(x^2 + 5x + 4)^2}$ M1A1A1

$= \dfrac{10x^2 - 40}{(x^2 + 5x + 4)^2} \quad \left(= \dfrac{10(x-2)(x+2)}{(x^2 + 5x + 4)^2} \right)$ A1

$f'(x) = 0 \Rightarrow x = \pm 2$ M1

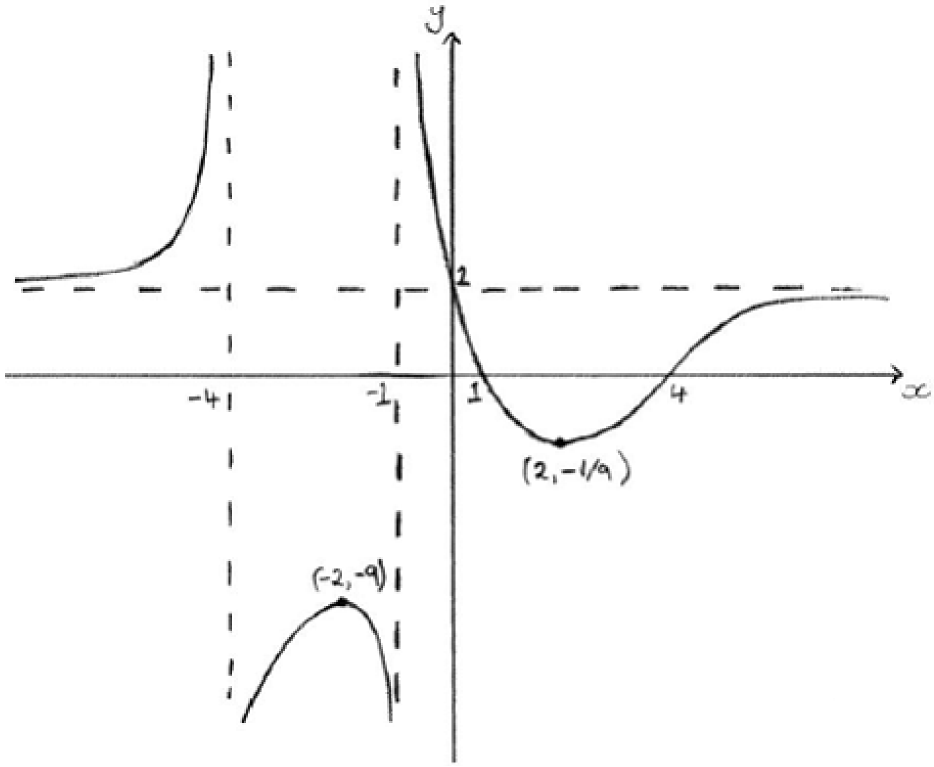
so the points under consideration are $(-2, -9)$ and $\left(2, -\dfrac{1}{9}\right)$ A1A1

looking at the sign either side of the points (or attempt to find $f''(x)$) M1
e.g. if $x = -2^-$ then $(x - 2)(x + 2) > 0$ and if $x = -2^+$ then $(x - 2)(x + 2) < 0$, therefore $(-2, -9)$ is a maximum A1

(ii) e.g. if $x = 2^-$ then $(x - 2)(x + 2) < 0$ and if $x = 2^+$ then $(x - 2)(x + 2) > 0$, therefore $\left(2, -\dfrac{1}{9}\right)$ is a minimum A1

Note: Candidates may find the minimum first.

(d)



A3

Note: Award A1 for each branch consistent with and including the features found in previous parts.

(e) one A1

[20]

32. Solve the equation $4^{x-1} = 2^x + 8$.

(Total 5 marks)

$2^{2x-2} = 2^x + 8$ (M1)

$\frac{1}{4}2^{2x} = 2^x + 8$ (A1)

$2^{2x} - 4 \times 2^x - 32 = 0$ A1

$(2^x - 8)(2^x + 4) = 0$ (M1)

$2^x = 8 \Rightarrow x = 3$ A1

Notes: Do not award final A1 if more than 1 solution is given.

[5]

33. Two players, A and B, alternately throw a fair six–sided dice, with A starting, until one of them obtains a six. Find the probability that A obtains the first six.

(Total 7 marks)

P (six in first throw) = $\frac{1}{6}$ (A1)

P (six in third throw) = $\frac{25}{36} \times \frac{1}{6}$ (M1)(A1)

P (six in fifth throw) = $\left(\frac{25}{36}\right)^2 \times \frac{1}{6}$

P(A obtains first six) = $\frac{1}{6} + \frac{25}{36} \times \frac{1}{6} + \left(\frac{25}{36}\right)^2 \times \frac{1}{6} + \ldots$ (M1)

recognizing that the common ratio is $\frac{25}{36}$ (A1)

P(A obtains first six) = $\frac{\frac{1}{6}}{1 - \frac{25}{36}}$ (by summing the infinite GP) M1

$= \frac{6}{11}$ A1

[7]

34. (a) Show that $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(2)

(b) Hence, or otherwise, find the value of arctan (2) + arctan (3).

(3)

(Total 5 marks)

(a) **METHOD 1**

let $x = \arctan\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ and $y = \arctan\frac{1}{3} \Rightarrow \tan y = \frac{1}{3}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$ *M1*

so, $x + y = \arctan 1 = \frac{\pi}{4}$ *A1AG*

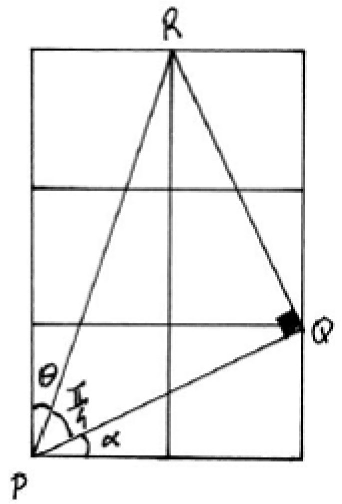
METHOD 2

for $x, y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ *M1*

so, $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \frac{\pi}{4}$ *A1AG*

METHOD 3

an appropriate sketch *M1*
e.g.



correct reasoning leading to $\frac{\pi}{4}$ *R1AG*

(b) **METHOD 1**

$\arctan(2) + \arctan(3) = \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) + \frac{\pi}{2} - \arctan\left(\frac{1}{3}\right)$ (M1)

$= \pi - \left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right)$ (A1)

Note: Only one of the previous two marks may be implied.

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ A1 N1

METHOD 2

let $x = \arctan 2 \Rightarrow \tan x = 2$ and $y = \arctan 3 \Rightarrow \tan y = 3$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2+3}{1-2\times 3} = -1$ (M1)

as $\frac{\pi}{4} < x < \frac{\pi}{2}\left(\text{accept } 0 < x < \frac{\pi}{2}\right)$

and $\frac{\pi}{4} < y < \frac{\pi}{2}\left(\text{accept } 0 < y < \frac{\pi}{2}\right)$

$\frac{\pi}{2} < x + y < \pi$ (accept $0 < x + y < \pi$) (R1)

Note: Only one of the previous two marks may be implied.

so, $x + y = \frac{3\pi}{4}$ A1 N1

METHOD 3

for $x, y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$ if $xy > 1$ (M1)

so, $\arctan 2 + \arctan 3 = \arctan\left(\frac{2+3}{1-2\times 3}\right) + \pi$ (A1)

Note: Only one of the previous two marks may be implied.

$= \frac{3\pi}{4}$ A1 N1

35. A function f is defined by $f(x) = \dfrac{2x-3}{x-1}$, $x \neq 1$.

(a) Find an expression for $f^{-1}(x)$.

(3)

(b) Solve the equation $\left|f^{-1}(x)\right| = 1 + f^{-1}(x)$.

(3)

(Total 6 marks)

(a) **Note:** Interchange of variables may take place at any stage.
for the inverse, solve for x in

$y = \dfrac{2x-3}{x-1}$

$y(x-1) = 2x-3$ M1

$yx - 2x = y - 3$

$x(y-2) = y-3$ (A1)

$x = \dfrac{y-3}{y-2}$

$\Rightarrow f^{-1}(x) = \dfrac{x-3}{x-2}$ $(x \neq 2)$ A1

Note: Do not award final A1 unless written in the form $f^{-1}(x) = \ldots$

(b) $\pm f^{-1}(x) = 1 + f^{-1}(x)$ leads to

$2\dfrac{x-3}{x-2} = -1$ (M1)A1

$x = \dfrac{8}{3}$ A1

[6]

36. Find the set of values of x for which $|x-1| > |2x-1|$.

(Total 4 marks)

$|x-1| > |2x-1| \Rightarrow (x-1)^2 > (2x-1)^2$ M1

$x^2 - 2x + 1 > 4x^2 - 4x + 1$

$3x^2 - 2x < 0$ A1

$0 < x < \dfrac{2}{3}$ A1A1 N2

Note: Award A1A0 for incorrect inequality signs.

37. The region enclosed between the curves $y = \sqrt{x}e^x$ and $y = e\sqrt{x}$ is rotated through 2π about the x-axis. Find the volume of the solid obtained.

(Total 7 marks)

$\sqrt{x}\,e^x = e\sqrt{x} \Rightarrow x = 0$ or 1 (A1)

attempt to find $\int y^2 dx$ M1

$V_1 = \pi\int_0^1 e^2 x dx$

$= \pi\left[\frac{1}{2}e^2x^2\right]_0^1$

$= \frac{\pi e^2}{2}$ A1

$V_2 = \pi\int_0^1 xe^{2x}dx$

$= \pi\left(\left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}dx\right)$ M1A1

Note: Award M1 for attempt to integrate by parts.

$= \frac{\pi e^2}{2} - \pi\left[\frac{1}{4}e^{2x}\right]_0^1$

finding difference of volumes M1

volume $= V_1 - V_2$

$= \pi\left[\frac{1}{4}e^{2x}\right]_0^1$

$= \frac{1}{4}\pi(e^2 - 1)$ A1

[7]

38. The function f is defined by $f(x) = \mathrm{e}^{x^2-2x-1.5}$.

(a) Find $f'(x)$.

(2)

(b) You are given that $y = \dfrac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of a.

(6)
(Total 8 marks)

(a) $\left(u = x^2 - 2x - 1.5; \dfrac{du}{dx} = 2x - 2\right)$

$\dfrac{df}{dx} = \dfrac{df}{du}\dfrac{du}{dx} = e^x(2x-2)$ (M1)

$= 2(x-1)e^{x^2-2x-1.5}$ A1

(b) $\dfrac{dy}{dx} = \dfrac{(x-1)\times 2(x-1)e^{x^2-2x-1.5} - 1\times e^{x^2-2x-1.5}}{(x-1)^2}$ M1A1

$= \dfrac{2x^2-4x+1}{(x+1)^2}e^{x^2-2x-1.5}$ (A1)

minimum occurs when $\dfrac{dy}{dx} = 0$ (M1)

$x = 1 \pm \sqrt{\dfrac{1}{2}}$ $\left(\text{accept } x = \dfrac{4 \pm \sqrt{8}}{4}\right)$ A1

$a = 1 + \sqrt{\dfrac{1}{2}}$ $\left(\text{accept } a = \dfrac{4+\sqrt{8}}{4}\right)$ R1

[8]

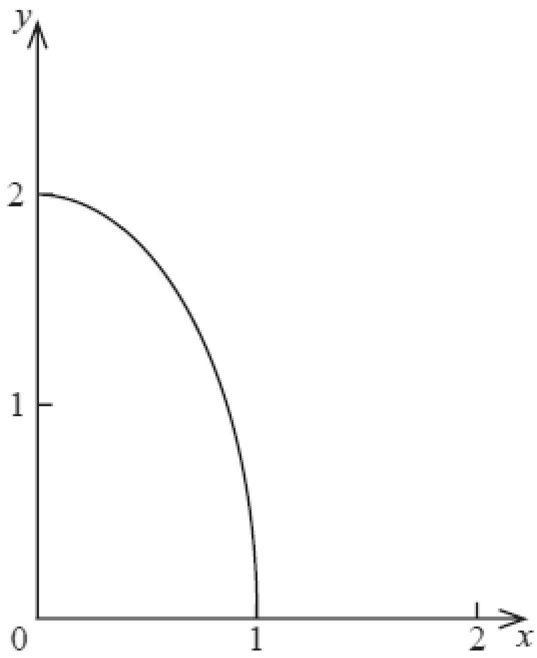
39. The mean of the first ten terms of an arithmetic sequence is 6. The mean of the first twenty terms of the arithmetic sequence is 16. Find the value of the 15th term of the sequence.

(Total 6 marks)

$5(2a + 9d) = 60$ (or $2a + 9d = 12$) M1A1
$10(2a + 19d) = 320$ (or $2a + 19d = 32$) A1
solve simultaneously to obtain M1
$a = -3$, $d = 2$ A1
the 15th term is $-3 + 14 \times 2 = 25$ A1

Note: FT the final A1 on the values found in the penultimate line.

40. Consider the part of the curve $4x^2 + y^2 = 4$ shown in the diagram below.



(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(3)

(b) Find the gradient of the tangent at the point $\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

(1)

(c) A bowl is formed by rotating this curve through 2π radians about the x-axis. Calculate the volume of this bowl.

(4)
(Total 8 marks)

(a) $8x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ M1A1

Note: Award M1A0 for $8x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{4x}{y}$ A1

(b) –4 A1

(c) $V = \int \pi y^2 \mathrm{d}x$ or equivalent M1

$V = \pi\int_0^1 (4 - 4x^2)\mathrm{d}x$ A1

$= \pi\left[4x - \dfrac{4}{3}x^3\right]_0^1$ A1

$= \dfrac{8\pi}{3}$ A1

Note: If it is correct except for the omission of π, award 2 marks.

[8]