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Mathematics: applications and interpretation HL formula booklet

For use during the course and in the examinations
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Contents

Topic 1: Number and algebra – HL

1.2	The nth term of an arithmetic sequence The sum of n terms of an arithmetic sequence	$u_n = u_1 + (n-1)d$ $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$; $S_n = \frac{n}{2}(u_1 + u_n)$
1.3	The nth term of a geometric sequence The sum of n terms of a finite geometric sequence	$u_n = u_1 r^{n-1}$ $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$
1.11	The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1-r}$, $\lvert r \rvert < 1$
1.4	Compound interest	$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$, where FV is the future value, PV is the present value, n is the number of years, k is the number of compounding periods per year, r% is the nominal annual rate of interest
1.5	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$
1.9	Laws of logarithms	$\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m \log_a x$ for $a, x, y > 0$
1.6	Percentage error	$\varepsilon = \left\lvert \frac{v_A - v_E}{v_E} \right\rvert \times 100\%$, where v_E is the exact value and v_A is the approximate value of v
1.12	Complex numbers Discriminant	$z = a + bi$ $\Delta = b^2 - 4ac$
1.13	Modulus-argument (polar) and exponential (Euler) form	$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\,\mathrm{cis}\,\theta$

1.14	Determinant of a 2×2 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = \lvert \boldsymbol{A} \rvert = ad - bc$
	Inverse of a 2×2 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$
1.15	Power formula for a matrix	$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$, where $\boldsymbol{P}$ is the matrix of eigenvectors and $\boldsymbol{D}$ is the diagonal matrix of eigenvalues

Topic 2: Functions – HL

Prior learning – HL	
Solutions of a quadratic equation	The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$

2.1	Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
	Gradient formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$
2.5	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$
2.9	Logistic function	$f(x) = \frac{L}{1 + C\mathrm{e}^{-kx}}$, $L, k, C > 0$

Topic 3: Geometry and trigonometry – HL

Prior learning – HL	
Area of a parallelogram	$A = bh$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(bh)$, where b is the base, h is the height
Area of a trapezoid	$A = \frac{1}{2}(a+b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a cuboid	$V = lwh$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Volume of prism	$V = Ah$, where A is the area of cross-section, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

3.1	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
	Volume of a right-pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base, h is the height

	Volume of a right cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
	Area of the curved surface of a cone	$A = \pi r l$, where r is the radius, l is the slant height
	Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
	Surface area of a sphere	$A = 4\pi r^2$, where r is the radius
3.2	Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
	Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
	Area of a triangle	$A = \frac{1}{2}ab\sin C$
3.4	Length of an arc	$l = \frac{\theta}{360} \times 2\pi r$, where θ is the angle measured in degrees, r is the radius
	Area of a sector	$A = \frac{\theta}{360} \times \pi r^2$, where θ is the angle measured in degrees, r is the radius
3.7	Length of an arc	$l = r\theta$, where r is the radius, θ is the angle measured in radians
	Area of a sector	$A = \frac{1}{2}r^2\theta$, where r is the radius, θ is the angle measured in radians
3.8	Identities	$\cos^2\theta + \sin^2\theta = 1$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

3.9	Transformation matrices	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$ $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, vertical stretch / stretch parallel to y-axis with a scale factor of k $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin ($\theta > 0$) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin ($\theta > 0$)
3.10	Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
3.11	Vector equation of a line Parametric form of the equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

3.13	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Angle between two vectors	$\cos\theta = \dfrac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
	Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a parallelogram	$A = \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram

Topic 4: Statistics and probability – HL

4.2	Interquartile range	$\text{IQR} = Q_3 - Q_1$
4.3	Mean, $\overline{x}$, of a set of data	$\overline{x} = \dfrac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$
4.5	Probability of an event A Complementary events	$\text{P}(A) = \dfrac{n(A)}{n(U)}$ $\text{P}(A) + \text{P}(A') = 1$
4.6	Combined events Mutually exclusive events Conditional probability Independent events	$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$
4.7	Expected value of a discrete random variable X	$\text{E}(X) = \sum_{i=1}^{k} x_i\ \text{P}(X = x_i)$
4.8	Binomial distribution $X \sim \text{B}(n, p)$ Mean Variance	 $\text{E}(X) = np$ $\text{Var}(X) = np(1-p)$

4.14	Linear transformation of a single random variable	$\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$ $\mathrm{Var}(aX+b)=a^2\,\mathrm{Var}(X)$
	Linear combinations of n independent random variables, $X_1, X_2, ..., X_n$	$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm ... \pm a_nX_n)=a_1\mathrm{E}(X_1)\pm a_2\mathrm{E}(X_2)\pm ... \pm a_n\mathrm{E}(X_n)$ $\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm ... \pm a_nX_n)$ $=a_1^2\,\mathrm{Var}(X_1)+a_2^2\,\mathrm{Var}(X_2)+...+a_n^2\,\mathrm{Var}(X_n)$
	Sample statistics	
	Unbiased estimate of population variance s_{n-1}^2	$s_{n-1}^2=\dfrac{n}{n-1}s_n^2$
4.17	Poisson distribution $X \sim \mathrm{Po}(m)$	
	Mean	$\mathrm{E}(X)=m$
	Variance	$\mathrm{Var}(X)=m$
4.19	Transition matrices	$\boldsymbol{T}^n\boldsymbol{s}_0=\boldsymbol{s}_n$, where $\boldsymbol{s}_0$ is the initial state

Topic 5: Calculus – HL

5.3	Derivative of x^n	$f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$
5.9	Derivative of $\sin x$	$f(x)=\sin x \Rightarrow f'(x)=\cos x$
	Derivative of $\cos x$	$f(x)=\cos x \Rightarrow f'(x)=-\sin x$
	Derivative of $\tan x$	$f(x)=\tan x \Rightarrow f'(x)=\dfrac{1}{\cos^2 x}$
	Derivative of e^x	$f(x)=\mathrm{e}^x \Rightarrow f'(x)=\mathrm{e}^x$
	Derivative of $\ln x$	$f(x)=\ln x \Rightarrow f'(x)=\dfrac{1}{x}$
	Chain rule	$y=g(u)$, where $u=f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Product rule	$y=uv \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=u\dfrac{\mathrm{d}v}{\mathrm{d}x}+v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Quotient rule	$y=\dfrac{u}{v} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x}-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
5.5	Integral of x^n	$\int x^n\,\mathrm{d}x=\dfrac{x^{n+1}}{n+1}+C, \quad n\neq -1$
5.8	The trapezoidal rule	$\int_a^b y\,\mathrm{d}x \approx \dfrac{1}{2}h\left((y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\right)$, where $h=\dfrac{b-a}{n}$

5.11	Standard integrals	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ $\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$ $\int e^x\,dx = e^x + C$
5.12	Area of region enclosed by a curve and x or y-axes Volume of revolution about x or y-axes	$A = \int_a^b \lvert y\rvert\,dx$ or $A = \int_a^b \lvert x\rvert\,dy$ $V = \int_a^b \pi y^2\,dx$ or $V = \int_a^b \pi x^2\,dy$
5.13	Acceleration Distance travelled from t_1 to t_2 Displacement from t_1 to t_2	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$ distance $= \int_{t_1}^{t_2} \lvert v(t)\rvert\,dt$ displacement $= \int_{t_1}^{t_2} v(t)\,dt$
5.16	Euler's method Euler's method for coupled systems	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant (step length) $x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$ where h is a constant (step length)
5.17	Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$