# Complex numbers - exam questions [49 marks]

**1.** [Maximum mark: 7] SPM.1.AHL.TZ0.15

Let $w = ae^{\frac{\pi}{4}i}$, where $a \in \mathbb{R}^+$.

for $a = 2$,

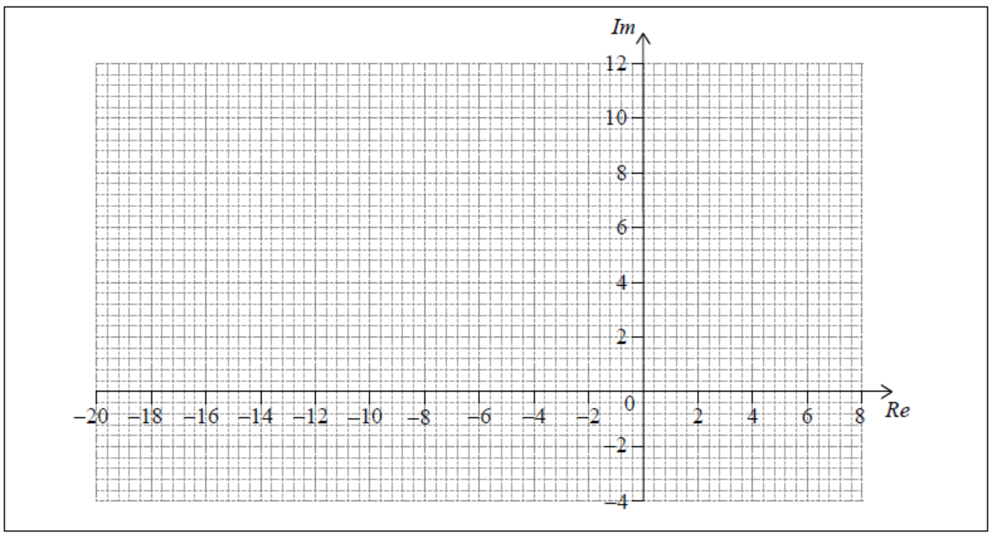
(a.i) find the values of $w^2$, $w^3$, and $w^4$. [2]

Markscheme

$4e^{\frac{\pi}{2}i}, 8e^{\frac{3\pi}{4}i}, 16e^{\pi i}$ $(= 4i,\ -4\sqrt{2} + 4\sqrt{2}i,\ -16)$ ***(M1)A1***

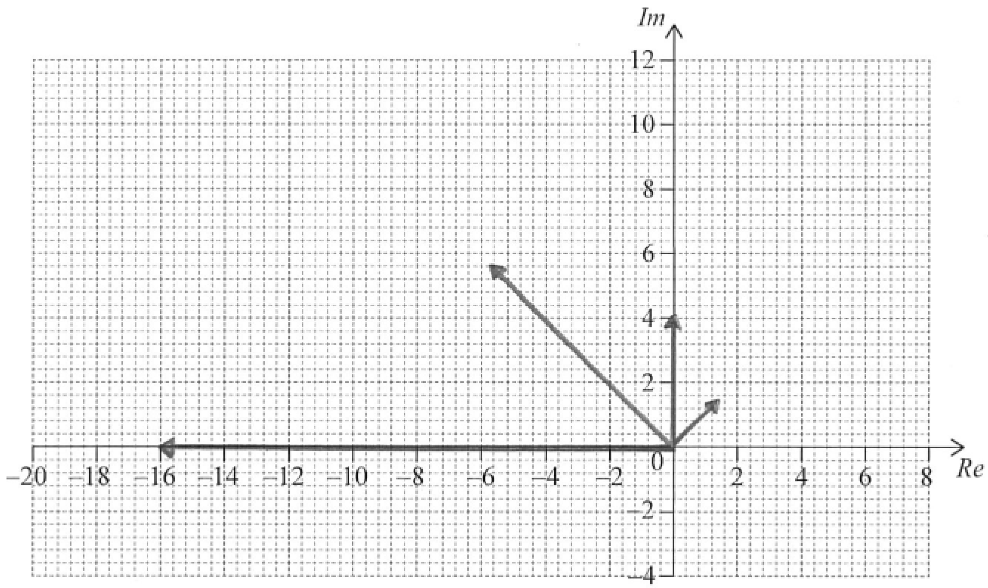
***[2 marks]***

(a.ii) draw $w$, $w^2$, $w^3$, and $w^4$ on the following Argand diagram.




[3]

Markscheme




***A3***

**Note:** Award ***A1*** for correct arguments, award ***A1*** for $4\mathrm{i}$ and −16 clearly indicated, award ***A1*** for | $w$ | < 4 and 4 < | $w^3$ | < 16.

***[3 marks]***

(b) Let $z = \frac{w}{2-\mathrm{i}}$.

Find the value of $a$ for which successive powers of $z$ lie on a circle. [2]

Markscheme

$2^2 + 1^2 = a^2$ ***M1***

$a = \sqrt{5}\ \ (= 2.24)$ ***A1***

***[2 marks]***

**2.** [Maximum mark: 5] EXN.1.AHL.TZ0.14

(a) Write down $2+5\mathrm{i}$ in exponential form. [2]

Markscheme

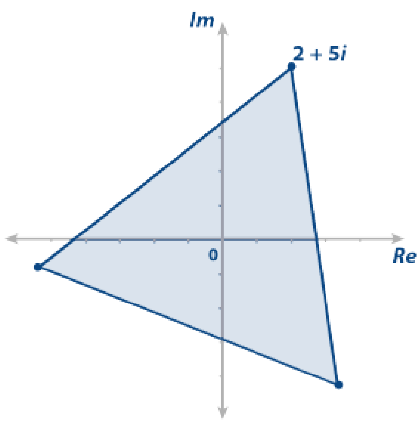
* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$5.385\ldots\mathrm{e}^{1.1902\ldots\mathrm{i}} \approx 5.39\mathrm{e}^{1.19\mathrm{i}}$ **A1A1**

**Note:** Accept equivalent answers: $5.39\mathrm{e}^{-5.09\mathrm{i}}$

**[2 marks]**

(b)




An equilateral triangle is to be drawn on the Argand plane with one of the vertices at the point corresponding to $2+5\mathrm{i}$ and all the vertices equidistant from $0$.

Find the points that correspond to the other two vertices. Give your answers in Cartesian form. [3]

Markscheme

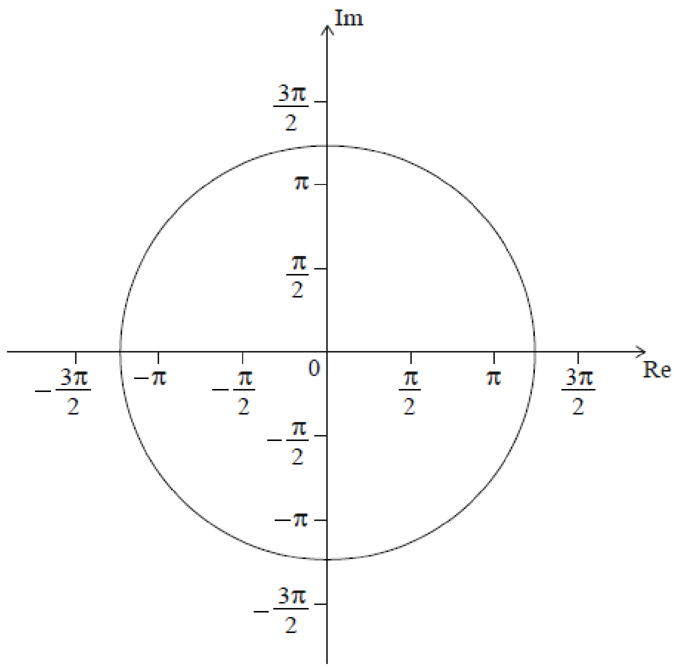
multiply by $\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$ **(M1)**

$-5.33-0.77\mathrm{i},\ 3.33-4.23\mathrm{i}$ **A1A1**

**[3 marks]**

**3.** [Maximum mark: 7] 22M.1.AHL.TZ1.10

The following Argand diagram shows a circle centre $0$ with a radius of $4$ units.



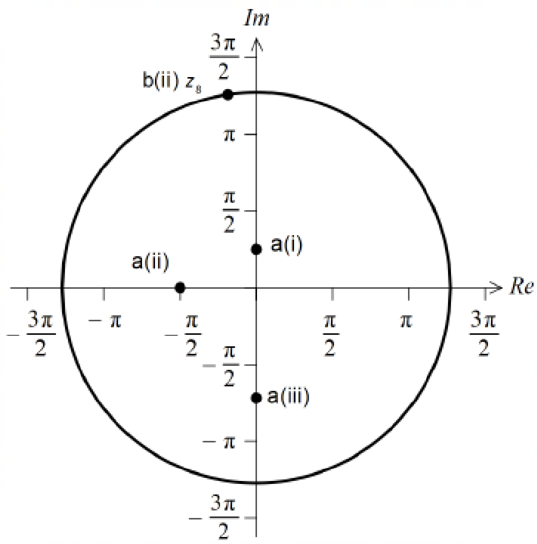

A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$z_\theta = \frac{1}{2}\theta e^{\theta i}$, where $\theta \geq 0$.

Plot on the Argand diagram the points corresponding to

(a.i) $\theta = \frac{\pi}{2}$. [1]

Markscheme



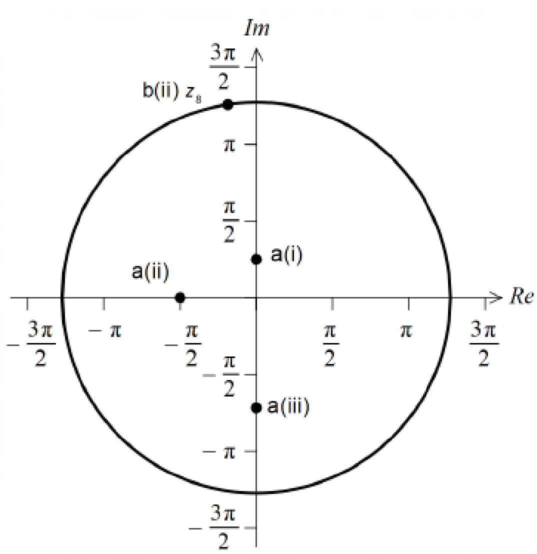

***A1***

**Note:** Award ***A1*** for correct modulus and ***A1*** for correct argument for part (a)(i), and ***A1*** for other two points correct. The points may not be labelled, and they may be shown by line segments.

***[1 mark]***

(a.ii) $\theta = \pi$. [1]

Markscheme



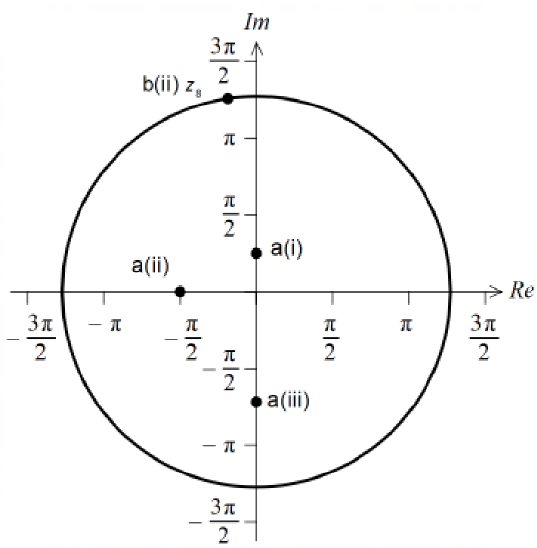

***A1***

**Note:** Award ***A1*** for correct modulus and ***A1*** for correct argument for part (a)(i), and ***A1*** for other two points correct. The points may not be labelled, and they may be shown by line segments.

***[1 mark]***

(a.iii) $\theta = \frac{3\pi}{2}$. [1]

Markscheme




***A1***

**Note:** Award ***A1*** for correct modulus and ***A1*** for correct argument for part (a)(i), and ***A1*** for other two points correct. The points may not be labelled, and they may be shown by line segments.

***[1 mark]***

Consider the case where $|z_\theta| = 4$.

(b.i) Find this value of $\theta$. [2]

Markscheme

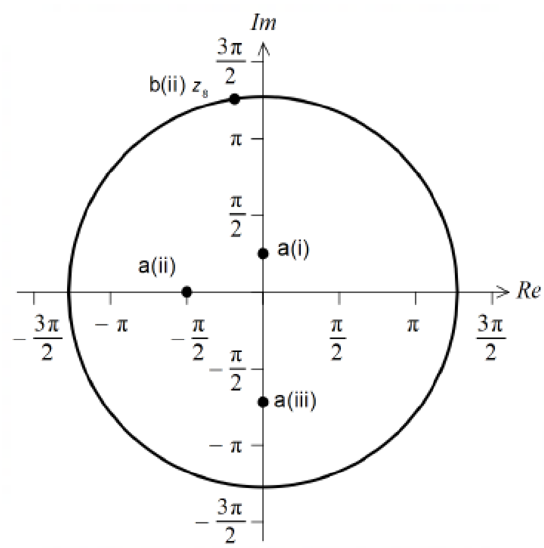
$\frac{1}{2}\theta = 4$ ***(M1)***

$\Rightarrow \theta = 8$ ***A1***

*[2 marks]*

(b.ii) For this value of $\theta$, plot the approximate position of $z_\theta$ on the Argand diagram. [2]

Markscheme




$z_8$ is shown in the diagram above ***A1A1***

**Note:** Award ***A1*** for a point plotted on the circle and ***A1*** for a point plotted in the second quadrant.

*[2 marks]*

**4.** [Maximum mark: 8] 21M.1.AHL.TZ1.9

Consider $w = \mathrm{i}z + 1$, where $w,\ z \in \mathbb{C}$.

Find $w$ when

(a.i) $z = 2\mathrm{i}$. [2]

Markscheme

$\mathrm{i}^2 = -1$ ***(M1)***

$w = -2 + 1 = -1$ ***A1***

***[2 marks]***

(a.ii) $z = 1 + \mathrm{i}$. [1]

Markscheme

$w = -1 + \mathrm{i} + 1 = \mathrm{i}$ ***A1***

***[1 mark]***

Point $z$ on the Argand diagram can be transformed to point $w$ by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

Markscheme

**EITHER**

rotation of $90^\circ$ (anticlockwise, centre at the origin) ***A1A1***

**Note:** Award ***A1*** for "rotation" and ***A1*** for "$90^\circ$".

followed by a translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ***A1***

**OR**

translation of $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ***A1***

followed by rotation of $90^\circ$ (anticlockwise, centre at the origin) ***A1A1***

**Note:** Award ***A1*** for "rotation" and A1 for "$90^\circ$".

***[3 marks]***

(c) Hence, or otherwise, find the value of $z$ when $w = 2 - \mathrm{i}$. [2]

Markscheme

**EITHER**

move $1$ to left to $1 - \mathrm{i}$ ***(M1)***

then rotate by $-90^\circ$ to

$-1 - \mathrm{i}$ ***A1***

**OR**

$\mathrm{i}z + 1 = 2 - \mathrm{i}$

$\mathrm{i}z = 1 - \mathrm{i}$

$z = \frac{1-\mathrm{i}}{\mathrm{i}}$ ***(M1)***

$-1 - \mathrm{i}$ ***A1***

***[2 marks]***

**5.** [Maximum mark: 8] 21M.1.AHL.TZ2.12

It is given that $z_1 = 3\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2\,\mathrm{cis}\left(\frac{n\pi}{16}\right),\ n \in \mathbb{Z}^+$.

In parts (a)(i) and (a)(ii), give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta},\ r \ge 0,\ \ -\pi < \theta \le \pi$.

(a.i) Find the value of ${z_1}^3$. [2]

Markscheme

${z_1}^3 = 27\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\ (= 27\mathrm{e}^{0.785398\ldots\mathrm{i}})$ ***A1A1***

**Note:** Award ***A1*** for $27$ and ***A1*** for the angle in the correct form.

***[2 marks]***

(a.ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [3]

Markscheme

$\left(\frac{z_1}{z_2}\right)^4 = \left(\frac{81}{16}\right)\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}\ (= 5.0625\mathrm{e}^{1.57079\ldots\mathrm{i}})$ ***A1A2***

**Note:** Award ***A1*** for $\frac{81}{16}$, ***A2*** for the angle in the correct form and ***A1*** for the angle in incorrect form e.g. $\mathrm{cis}\frac{\pi}{2}$ and/or $\frac{5\pi}{2}$. Award ***A1*** if $\mathrm{i}$ is given in place of $\mathrm{cis}\frac{\pi}{2}$.

***[3 marks]***

(b) Find the least value of $n$ such that $z_1 z_2 \in \mathbb{R}^+$. [3]

Markscheme

$z_1 z_2 = 6\,\mathrm{cis}\left(\frac{3\pi}{4} + \frac{n\pi}{16}\right)$ ***(M1)***

$= 6\,\mathrm{cis}\left(\frac{12\pi + n\pi}{16}\right)$

$12\pi + n\pi = 32\pi$ ***(M1)***

$n = 20$ ***A1***

*[3 marks]*

**6.** [Maximum mark: 14] 18M.1.AHL.TZ1.H_11

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

(a.i) Express $w^2$ and $w^3$ in modulus-argument form. [3]

Markscheme

$w^2 = 4\mathrm{cis}\left(\frac{2\pi}{3}\right);\ w^3 = 8\mathrm{cis}\left(\pi\right)$ ***(M1)A1A1***

**Note:** Accept Euler form.

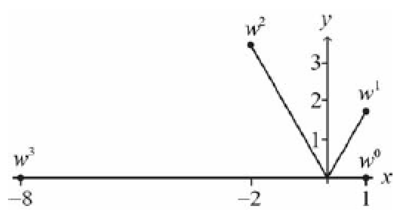
**Note:** ***M1*** can be awarded for either both correct moduli or both correct arguments.

**Note:** Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

***[3 marks]***

(a.ii) Sketch on an Argand diagram the points represented by $w^0$, $w^1$, $w^2$ and $w^3$. [2]

Markscheme




***A1A1***

***[2 marks]***

These four points form the vertices of a quadrilateral, *Q*.

(b) Show that the area of the quadrilateral *Q* is $\frac{21\sqrt{3}}{2}$. [3]

Markscheme

use of area = $\frac{1}{2}ab\ \sin C$ ***M1***

$\frac{1}{2} \times 1 \times 2 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 2 \times 4 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 4 \times 8 \times \sin\frac{\pi}{3}$ ***A1A1***

**Note:** Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

**Note:** Other methods of splitting the area may receive full marks.

***[3 marks]***

(c) Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right),\ n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2,\ \ldots,\ z^n$ form the vertices of a polygon $P_n$.

Show that the area of the polygon $P_n$ can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a,\ b \in \mathbb{R}$. [6]

Markscheme

$\frac{1}{2} \times 2^0 \times 2^1 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^1 \times 2^2 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^2 \times 2^3 \times \sin\frac{\pi}{n} + \ldots + \frac{1}{2} \times 2^{n-1} \times 2^n \times \sin\frac{\pi}{n}$ ***M1A1***

**Note:** Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n} \times \left(2^0 + 2^2 + 2^4 + \ldots + 2^{n-2}\right)$

identifying a geometric series with common ratio $2^2 (= 4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4} \times \sin\frac{\pi}{n}$ ***M1***

**Note:** Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}\left(4^n - 1\right)\sin\frac{\pi}{n}$ ***A1***

***[6 marks]***