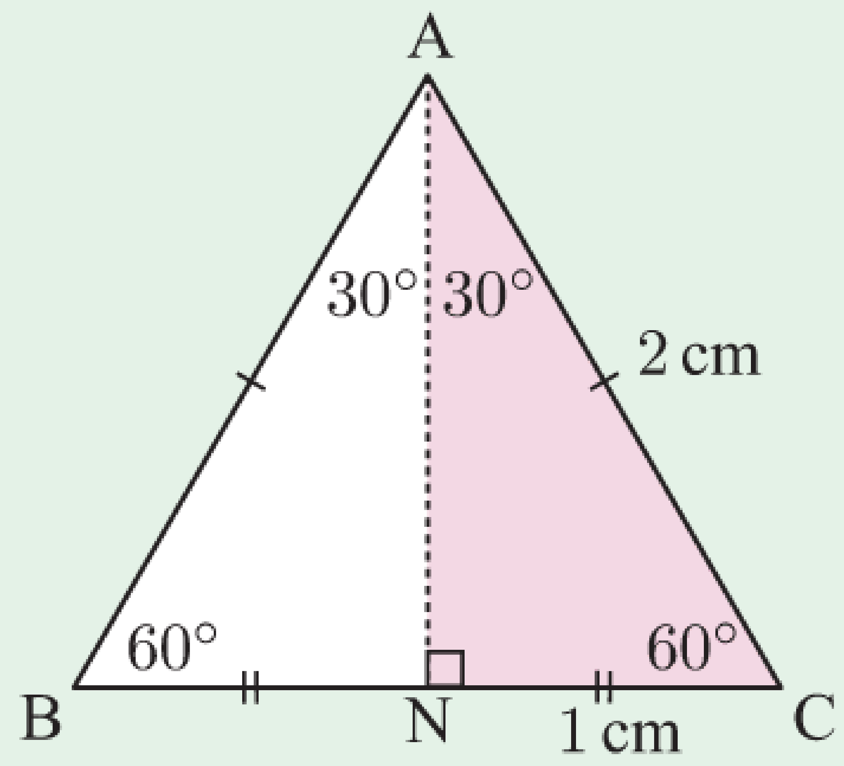
Chapter 8

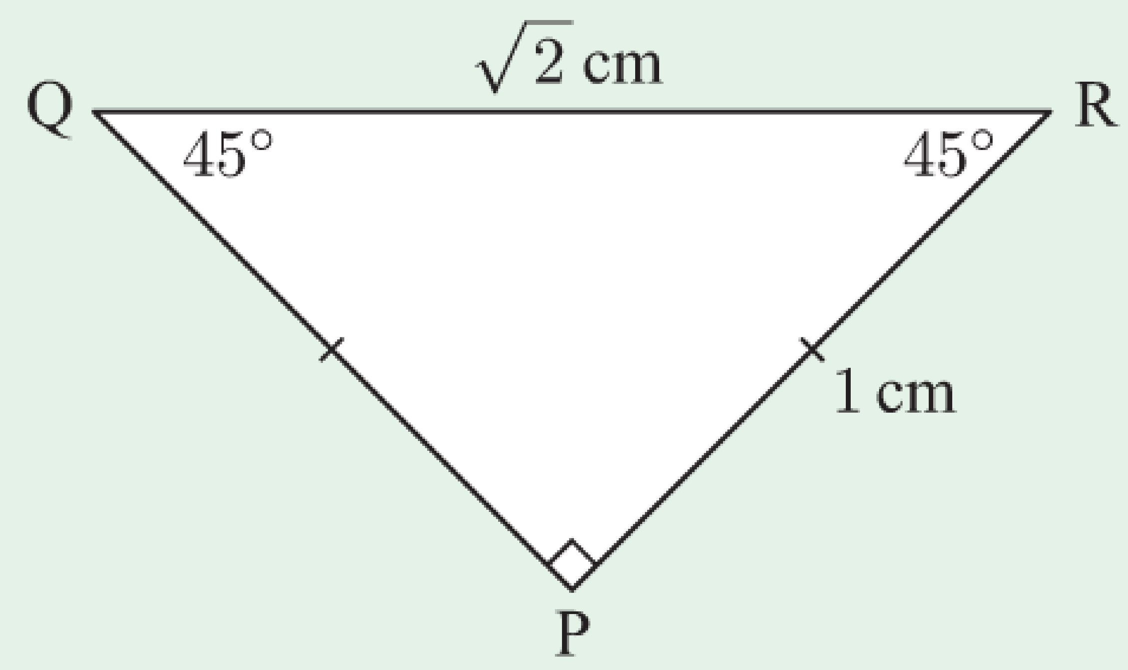
The unit circle and radian measure

Contents:

OPENING PROBLEM

Consider the triangles below:





Things to think about:

a Triangle ABC is an equilateral triangle with sides 2 cm long. Altitude [AN] bisects side [BC] and the vertical angle BAC.
Can you use this figure to find:
i $\sin 30°$ **ii** $\cos 60°$ **iii** $\cos 30°$ **iv** $\sin 60°$?

b Triangle PQR is a right angled isosceles triangle with hypotenuse $\sqrt{2}$ cm long.
Can you use this figure to find:
i $\cos 45°$ **ii** $\sin 45°$ **iii** $\tan 45°$?

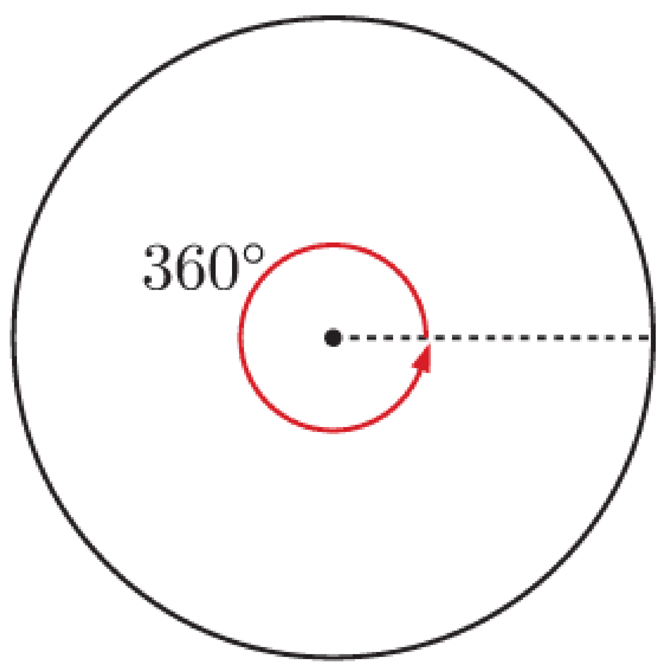
In this Chapter we build on our knowledge of angles and trigonometry. We consider:

- **radian** measure as an alternative to degrees
- the **unit circle** which helps us give meaning to the trigonometric ratios for *any* angle.

A RADIAN MEASURE

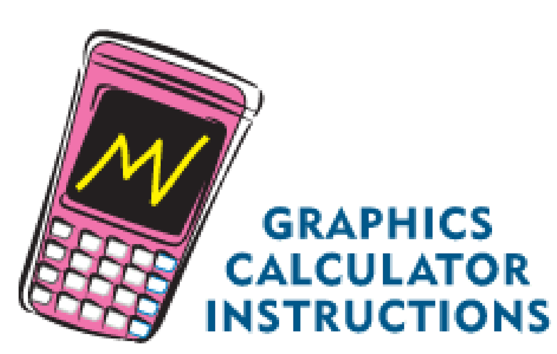
DEGREE MEASUREMENT OF ANGLES

We have seen previously that one full revolution makes an angle of $360°$, and the angle on a straight line is $180°$. Hence, one **degree**, $1°$, can be defined as $\frac{1}{360}$th of one full revolution. This measure of angle is commonly used by surveyors and architects.



For greater accuracy we define one **minute**, $1'$, as $\frac{1}{60}$th of one degree and one **second**, $1''$, as $\frac{1}{60}$th of one minute. Obviously a minute and a second are very small angles.

Most graphics calculators can convert fractions of angles measured in degrees into minutes and seconds. This is also useful for converting fractions of hours into minutes and seconds for time measurement, as one minute is $\frac{1}{60}$th of one hour, and one second is $\frac{1}{60}$th of one minute.



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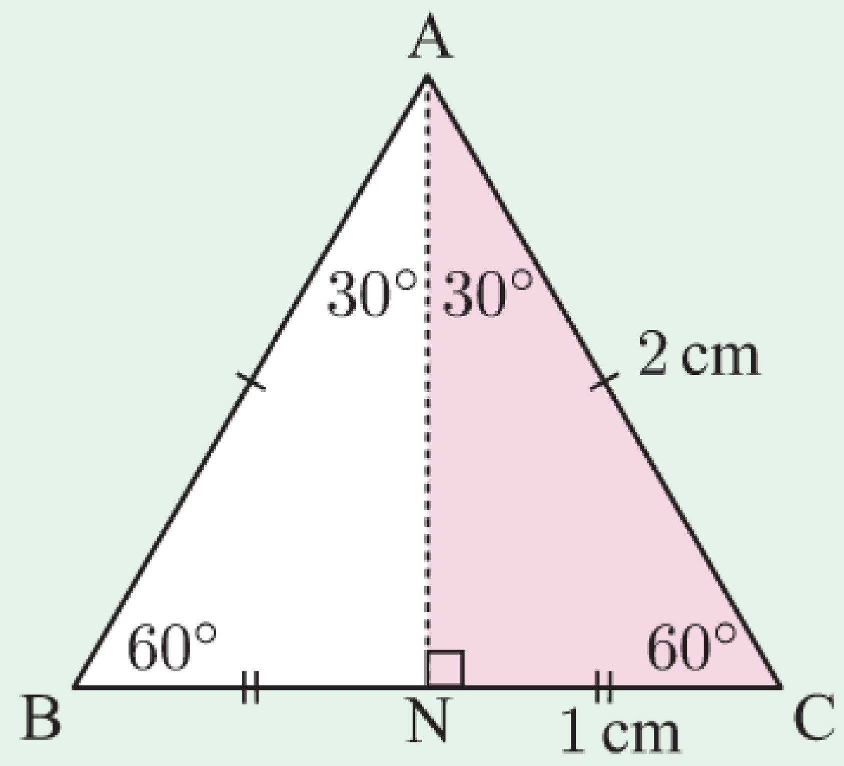

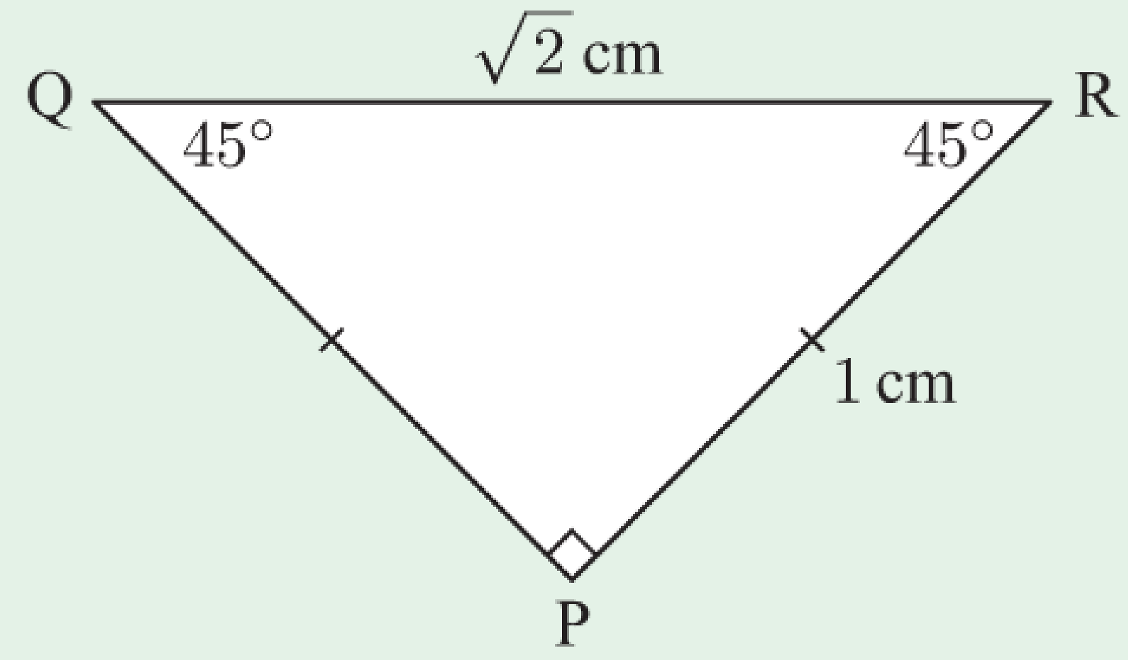

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A RADIAN MEASURE

DEGREE MEASUREMENT OF ANGLES

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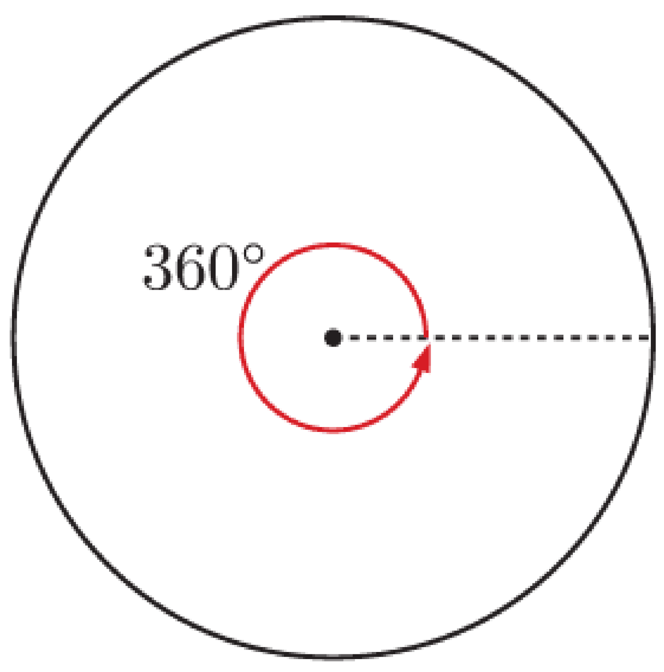

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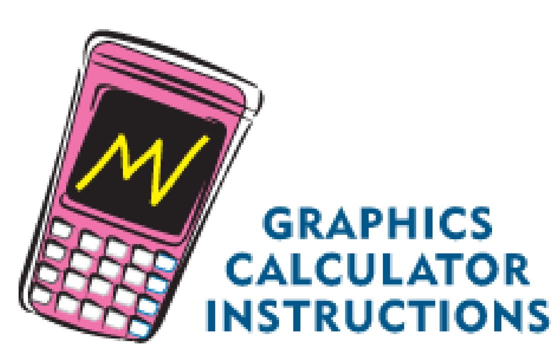


EXERCISE 8A

1 Convert to radians, in terms of π:

a $90°$ **b** $60°$ **c** $30°$ **d** $18°$ **e** $9°$
f $135°$ **g** $225°$ **h** $270°$ **i** $360°$ **j** $720°$
k $315°$ **l** $540°$ **m** $36°$ **n** $80°$ **o** $230°$

2 Convert to radians, correct to 3 significant figures:

a $36.7°$ **b** $137.2°$ **c** $317.9°$ **d** $219.6°$ **e** $396.7°$

Example 2 **Self Tutor**

Convert to degrees:

a $\frac{5\pi}{6}$ **b** 0.638 radians.

a $\frac{5\pi}{6}$
$= \left(\frac{5\pi}{6} \times \frac{180}{\pi}\right)^\circ$
$= 150°$

b 0.638 radians
$= \left(0.638 \times \frac{180}{\pi}\right)^\circ$
$\approx 36.6°$

3 Convert to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{7\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert to degrees, correct to 2 decimal places:

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete, giving answers in terms of π:

a

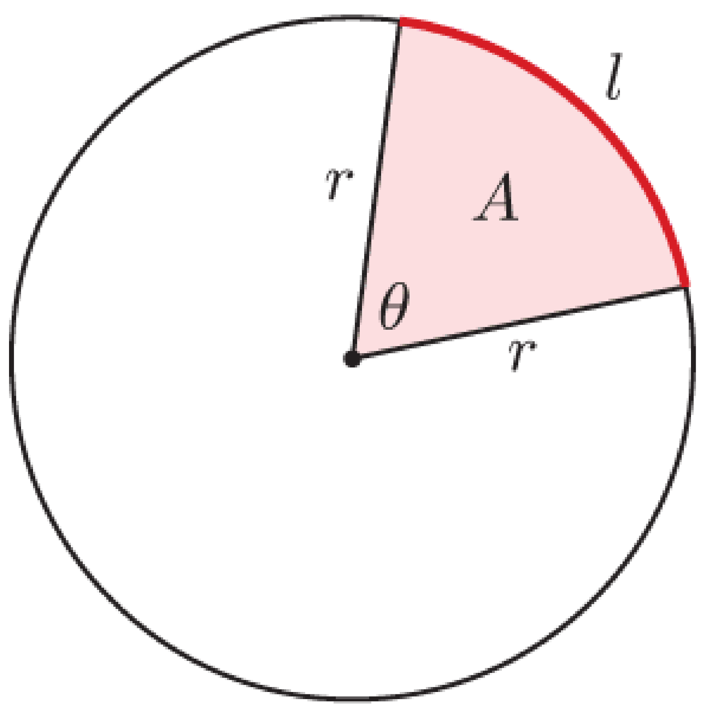
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

You should have previously seen formulae for the length of an arc and the area of a sector, for an angle given in degrees.



For a sector with radius r and angle θ given in *degrees*,

$$\text{arc length} \quad l = \frac{\theta}{360} \times 2\pi r$$

$$\text{area} \quad A = \frac{\theta}{360} \times \pi r^2$$

However, if the angle θ is measured in radians, the formulae become much simpler.

- θ measures how many times longer the arc length is than the radius.

$$\therefore \quad \theta = \frac{l}{r}$$

$$\therefore \quad l = \theta r$$

- There are 2π radians in a circle so

$$\text{area of sector} = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$\therefore \quad A = \frac{\theta}{2\pi} \times \pi r^2$$

$$\therefore \quad A = \tfrac{1}{2}\theta r^2$$

For a sector with radius r and angle θ given in *radians*:

- arc length $\boldsymbol{l = \theta r}$
- area $\boldsymbol{A = \tfrac{1}{2}\theta r^2}$

Example 3 **Self Tutor**

A sector has radius 12 cm and angle 3 radians. Find its:

a arc length **b** area

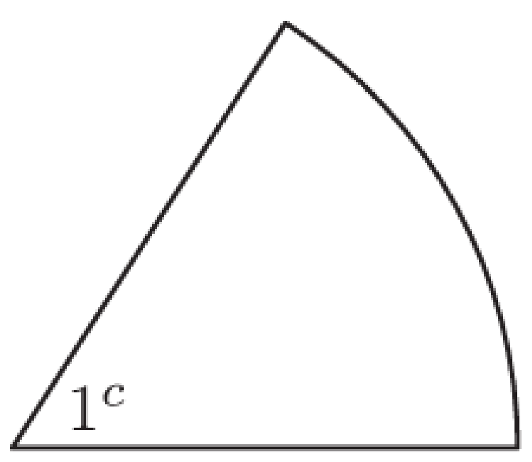
a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216 \text{ cm}^2$

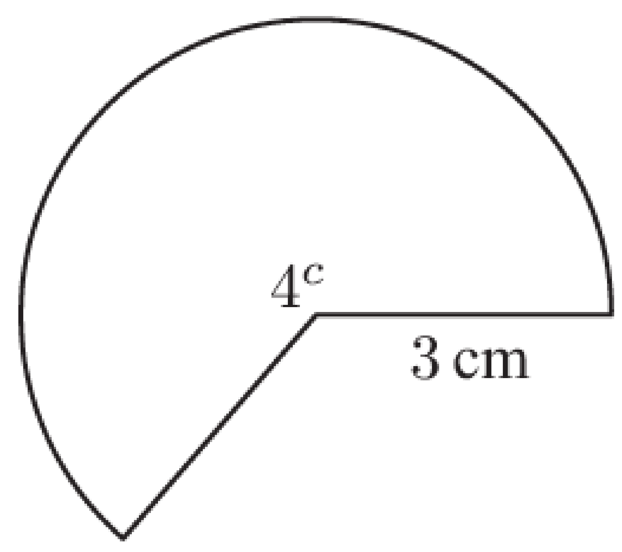
EXERCISE 8B

1 Find the arc length of each sector:

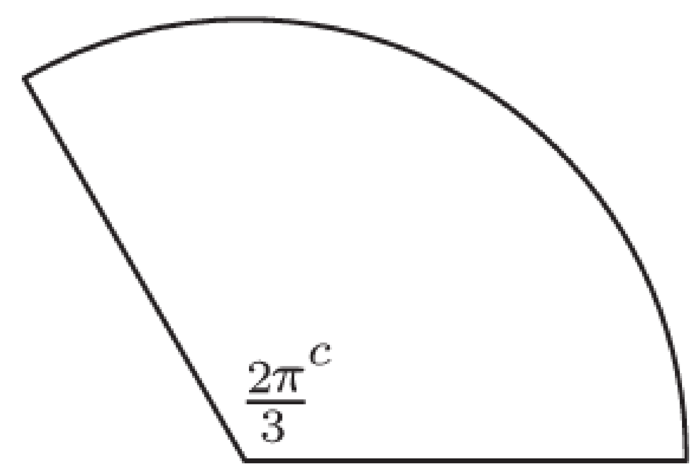
a



b

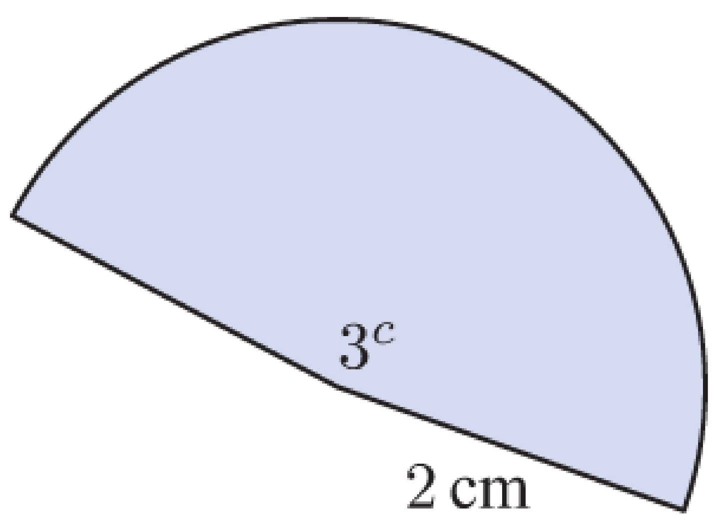


c



2 Find the area of each sector:

a



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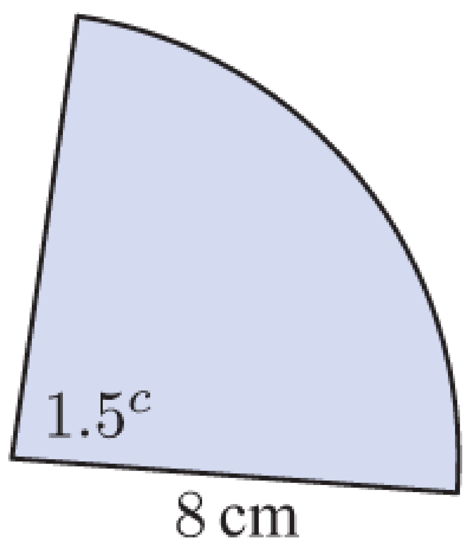

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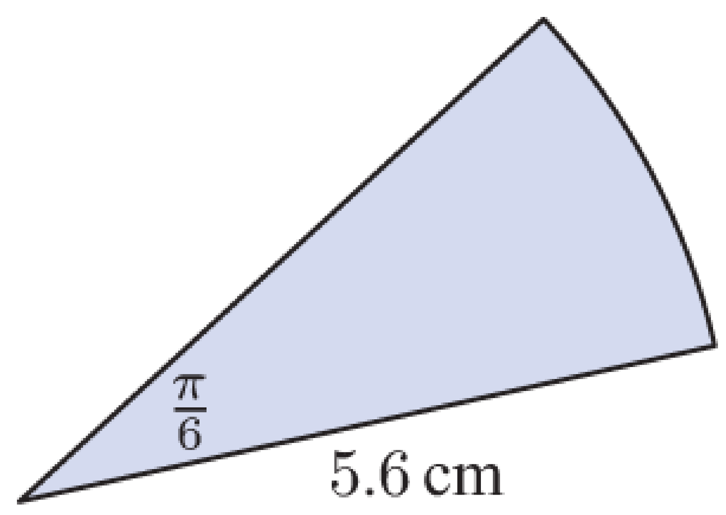


3 Find the arc length and area of a sector of a circle with:

a radius 9 cm and angle $\frac{7\pi}{4}$ **b** radius 4.93 cm and angle 4.67 radians.

Example 4 **Self Tutor**

A sector has radius 8.2 cm and arc length 12.3 cm. Find its:

a angle **b** area.

a $l = \theta r$ $\{\theta$ in radians$\}$

$$\therefore \quad \theta = \frac{l}{r} = \frac{12.3}{8.2} = 1.5 \text{ radians}$$

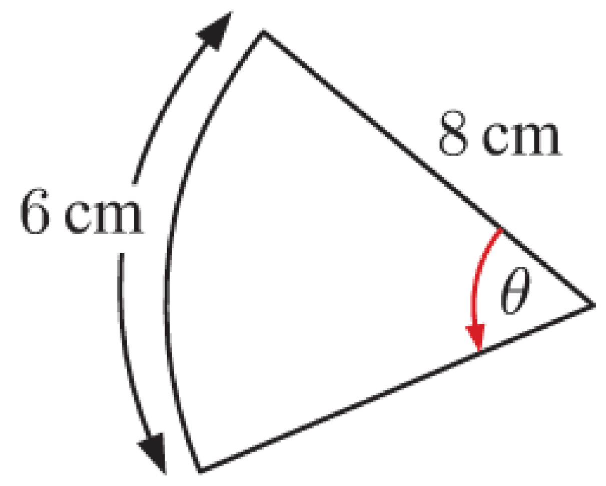
b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 1.5 \times 8.2^2$
$= 50.43 \text{ cm}^2$

4 Find, in radians, the angle of a sector of:

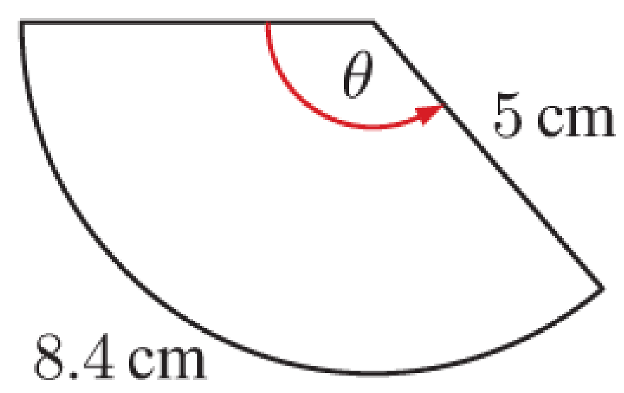
a radius 4.3 m and arc length 2.95 m **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

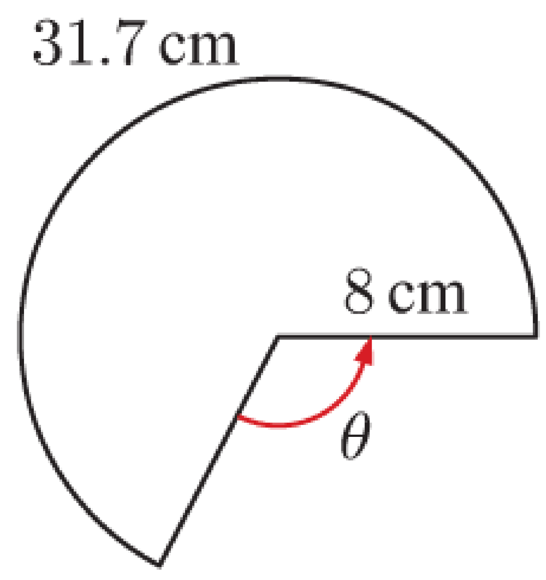
a



b



c



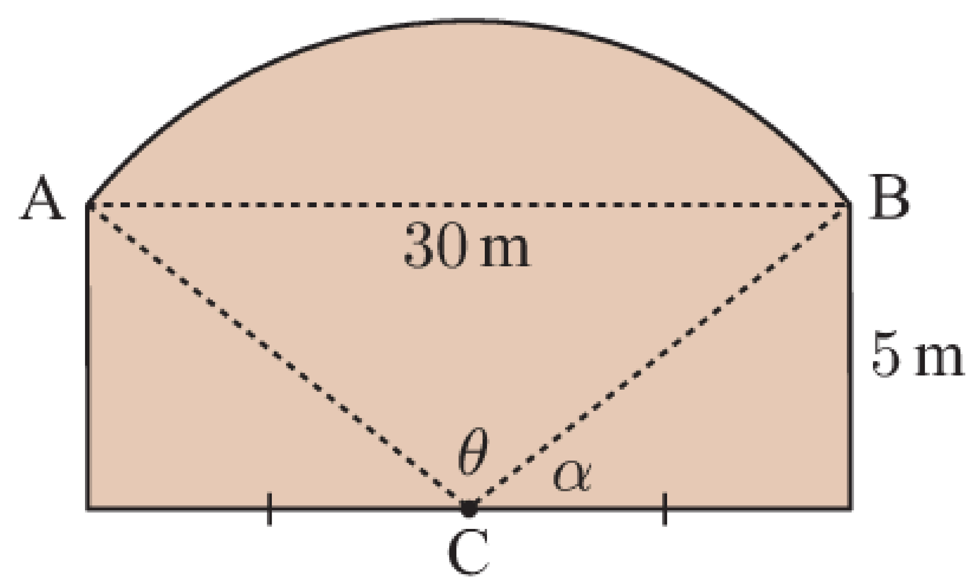
6 A sector has an angle of 1.88 radians and an arc length of 5.92 m. Find its:

a radius **b** area.

7 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find its:

a radius **b** perimeter.

8



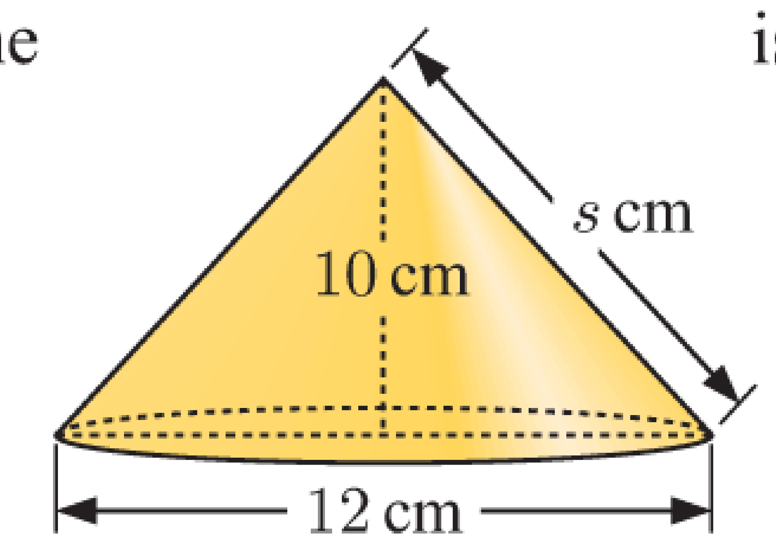
The end wall of a building has the shape illustrated, where the centre of arc AB is at C. Find:

a α in radians to 4 significant figures

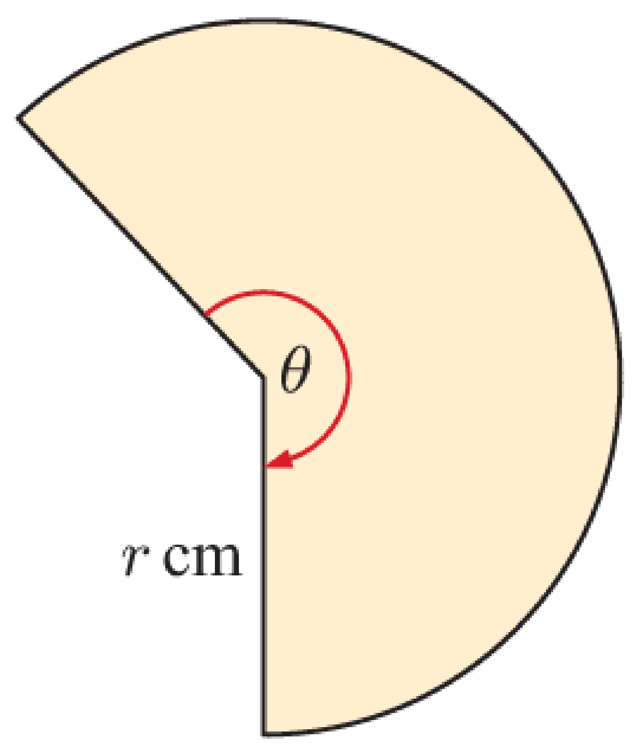
b θ in radians to 4 significant figures

c the area of the wall.

9 The cone



is made from this sector:

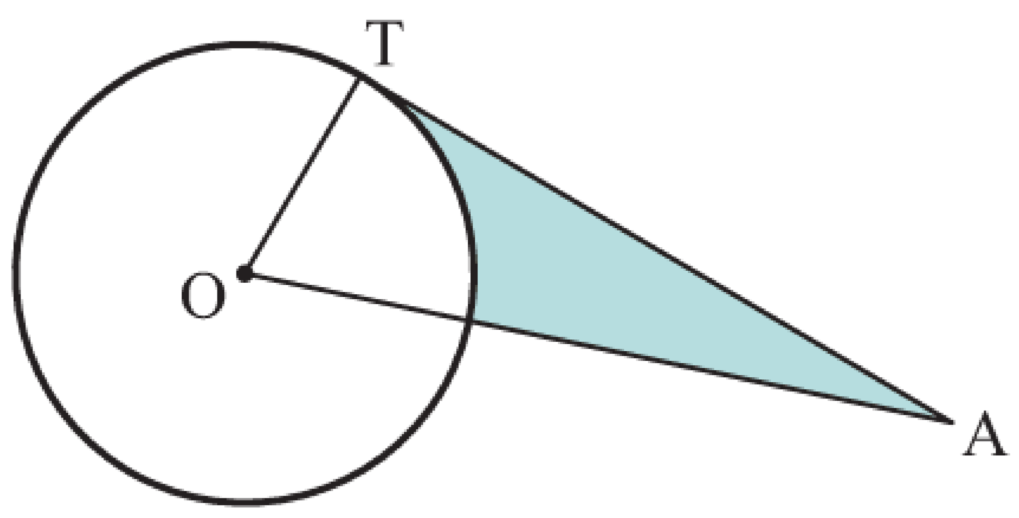


Find, correct to 3 significant figures:

a the slant length s cm **b** the value of r

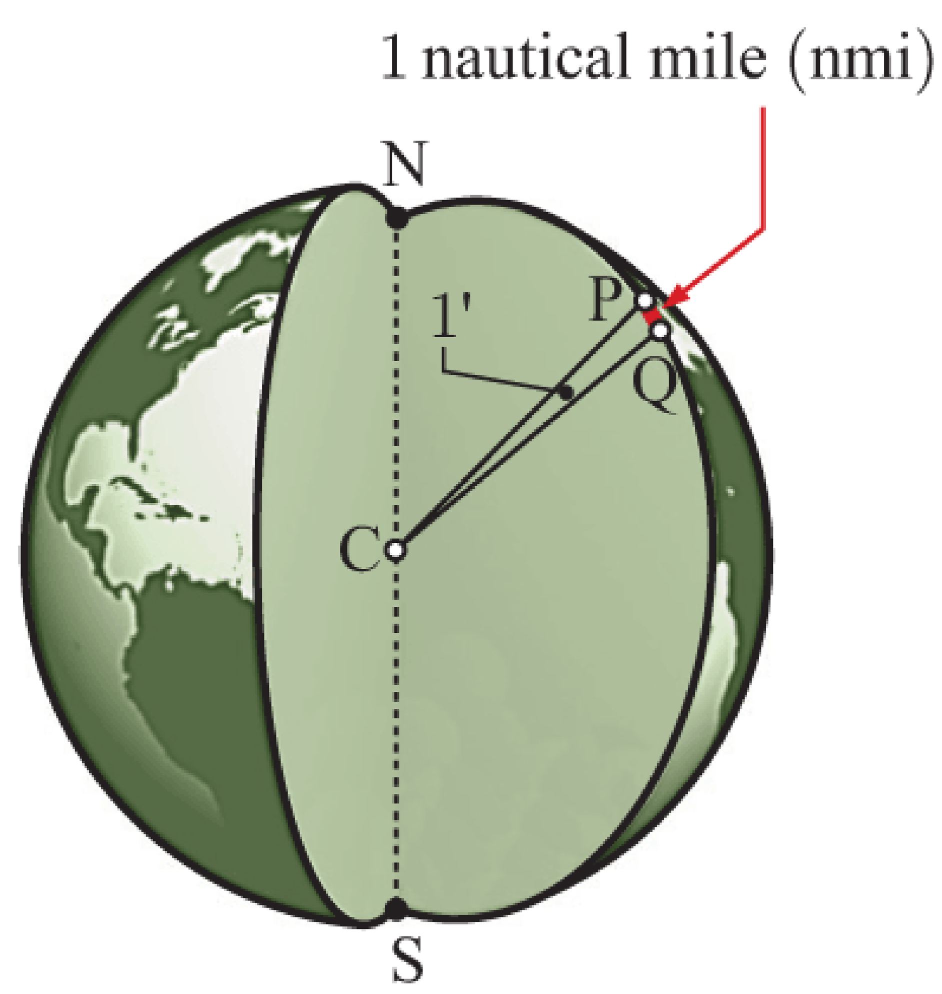
c the arc length of the sector **d** the sector angle θ in radians.

10



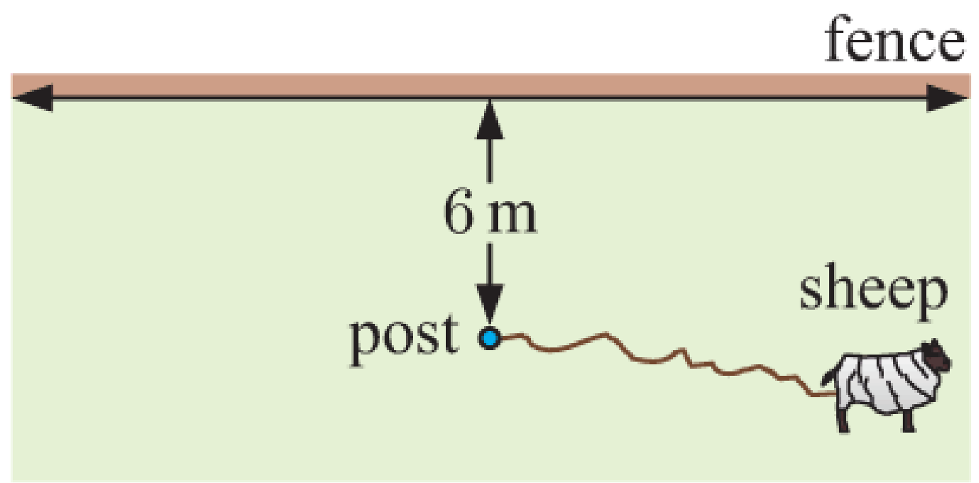
[AT] is a tangent to the given circle. OA = 13 cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$th of a degree) of the Great Circle arc measured from the centre of the Earth. A **knot** is a speed of 1 nautical mile per hour.



a Given that the radius of the Earth is 6370 km, show that 1 nmi is approximately 1.853 km.

b Calculate how long it would take a plane to fly 2130 km from Perth to Adelaide if the plane can fly at 480 knots.

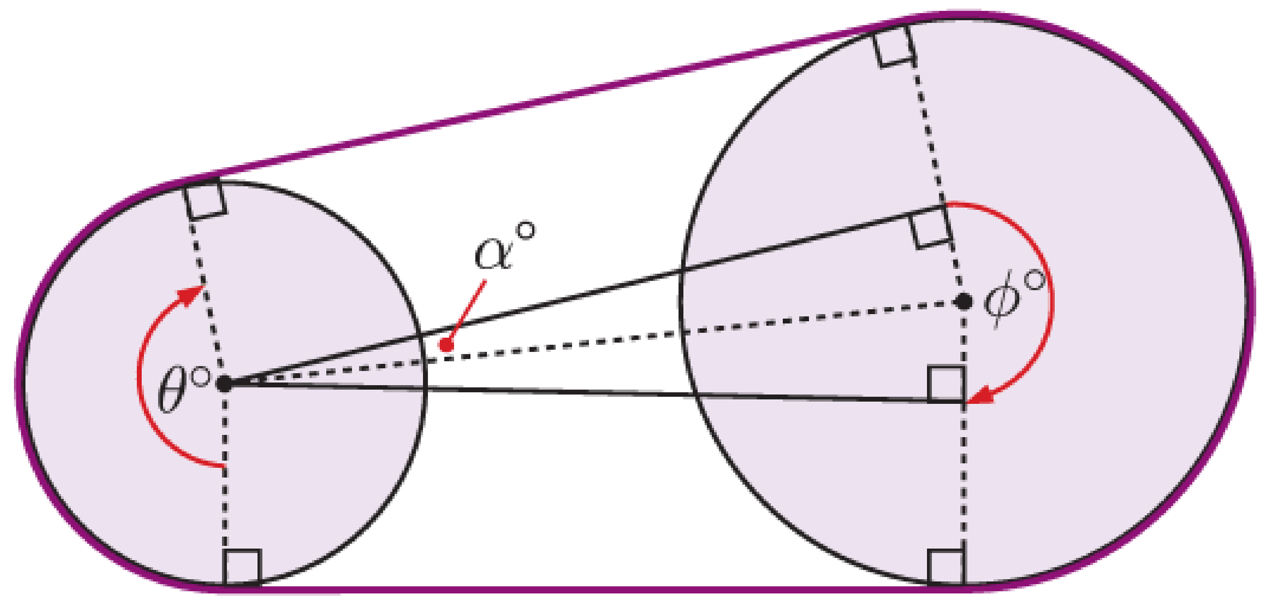
12 

A sheep is tethered to a post which is 6 m from a long fence. The length of the rope is 9 m. Find the area which the sheep can feed on.

13 A belt fits tightly around two pulleys with radii 4 cm and 6 cm respectively. The distance between their centres is 20 cm.
Find, correct to 4 significant figures:

a α **b** θ **c** ϕ

d the length of the belt.

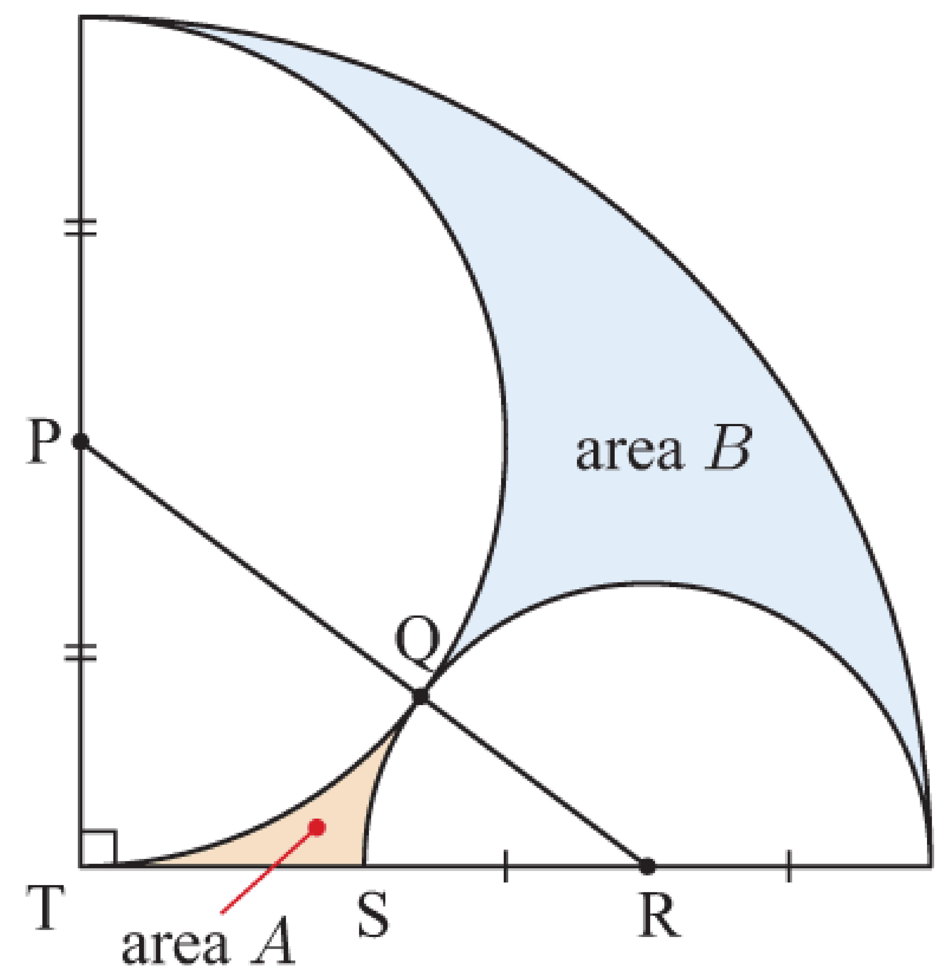


14 Two semi-circles touch each other within a quarter circle as shown. P, Q, and R are collinear. The radius of the quarter circle is 12 cm.

a Find the radius of the smaller semi-circle.

b Calculate the area of:

i A **ii** B.

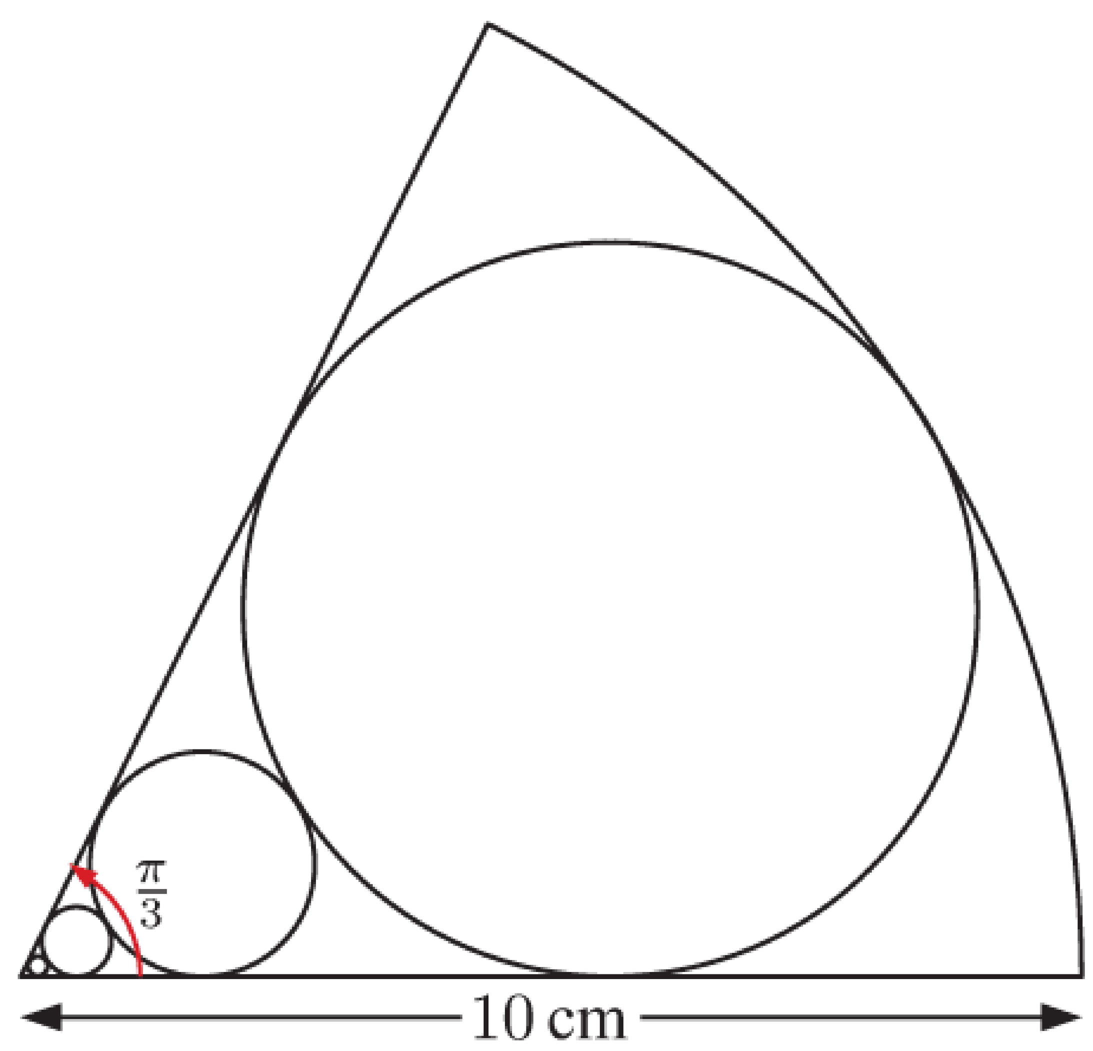


15 An infinite number of circles are drawn in a sector of a circle as shown.

a Show that the largest circle has radius $\frac{10}{3}$ cm.

b Find the total area of this infinite series of circles.

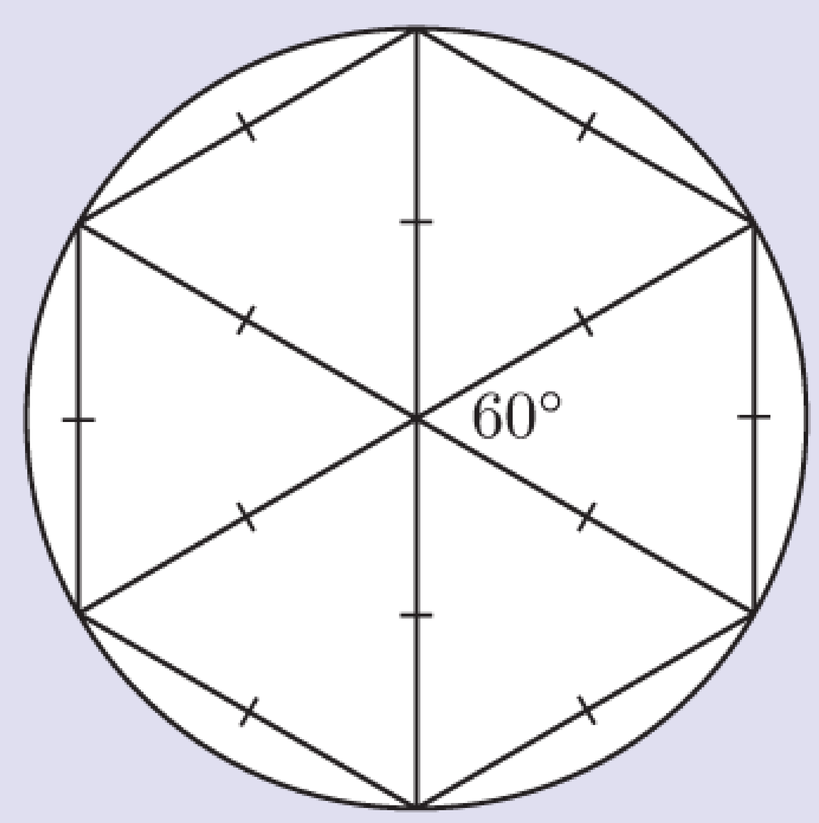
c What fraction of the sector is occupied by the circles?



THEORY OF KNOWLEDGE

There are several theories for why one complete turn was divided into 360 degrees:

- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.



By contrast, we have seen how radians are convenient in simplifying formulae which relate angles with distances and areas.

1 Which angle measure do you think is more:

a *practical* **b** *natural* **c** *mathematical*?

2 What other measures of angle are there, and for what purpose were they defined?

3 Which temperature scale, Celsius, Kelvin, or Fahrenheit, do you think is more:

a *practical* **b** *natural*?

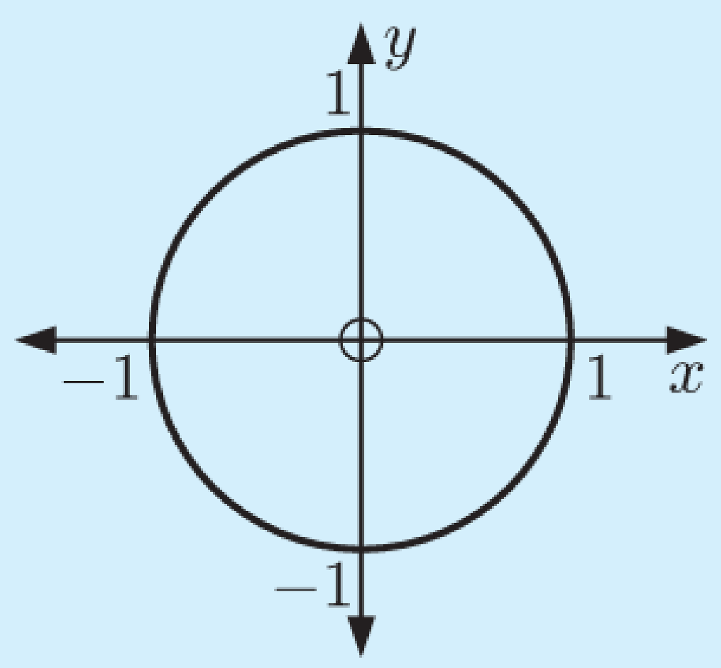
4 What other measures have we defined as a way of convenience?

5 What things are done differently around the world, but would be useful to globally standardise? For example, why are there different power voltages in different countries? Why have they not been standardised?

6 What things do we measure in a particular way simply for reasons of history rather than practical purpose?

C THE UNIT CIRCLE

The **unit circle** is the circle with centre $(0, 0)$ and radius 1 unit.

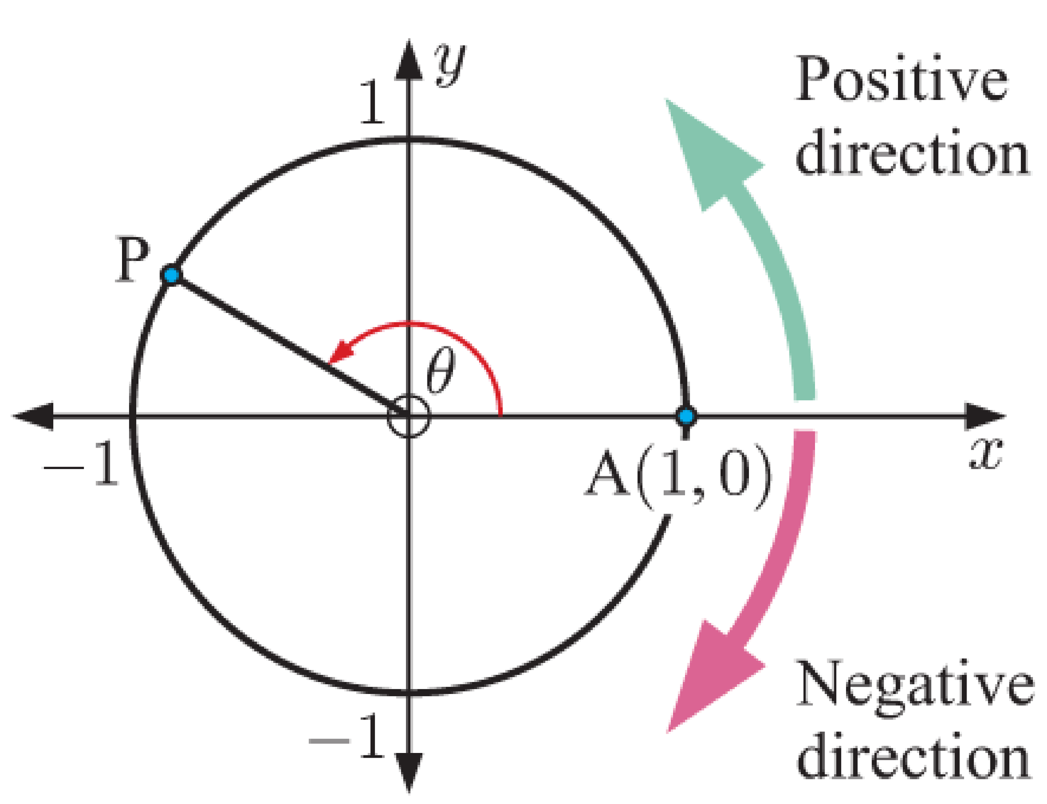


Applying the distance formula to a general point (x, y) on the circle, we find the equation of the unit circle is $x^2 + y^2 = 1$.

ANGLE MEASUREMENT

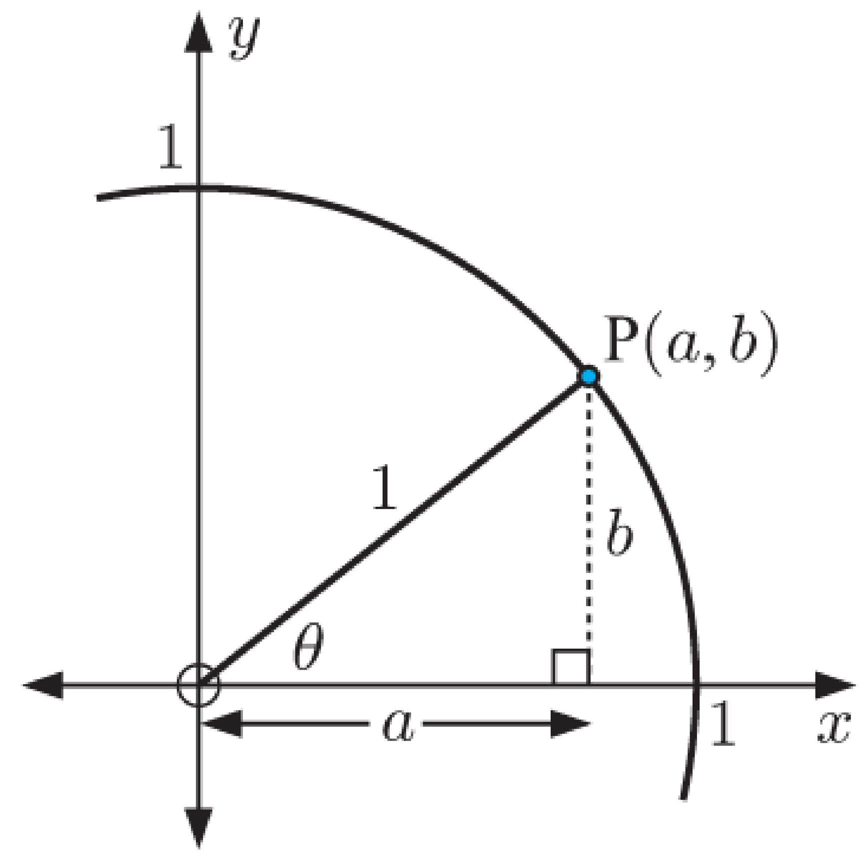
Suppose P lies anywhere on the unit circle, and A is $(1, 0)$. Let θ be the angle measured anticlockwise from [OA] on the positive x-axis.

θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



DEFINITION OF SINE AND COSINE

Consider a point $P(a, b)$ which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.



Using right angled triangle trigonometry:

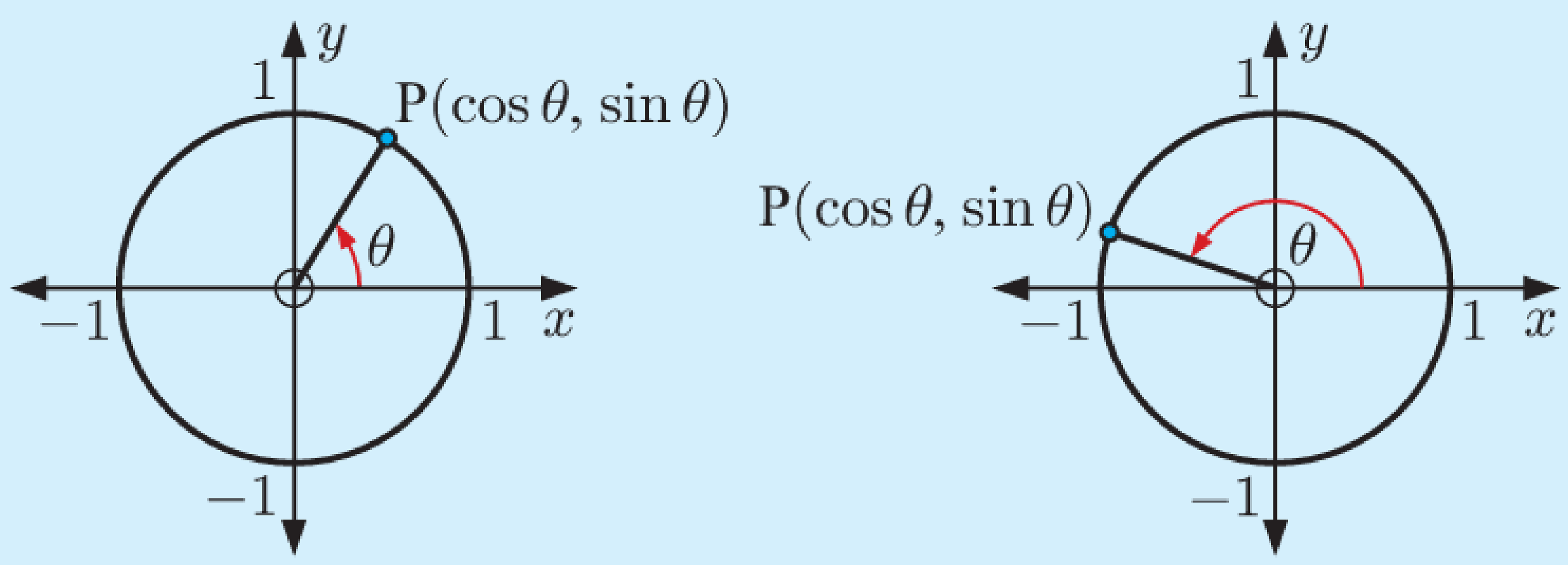
$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

More generally, we define:

If P is any point on the unit circle such that [OP] makes an angle θ measured anticlockwise from the positive x-axis:

- $\boldsymbol{\cos\theta}$ is the x-coordinate of P
- $\boldsymbol{\sin\theta}$ is the y-coordinate of P



For all points on the unit circle, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$. We therefore conclude:

For any angle θ:

- $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$
- $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

DEFINITION OF TANGENT

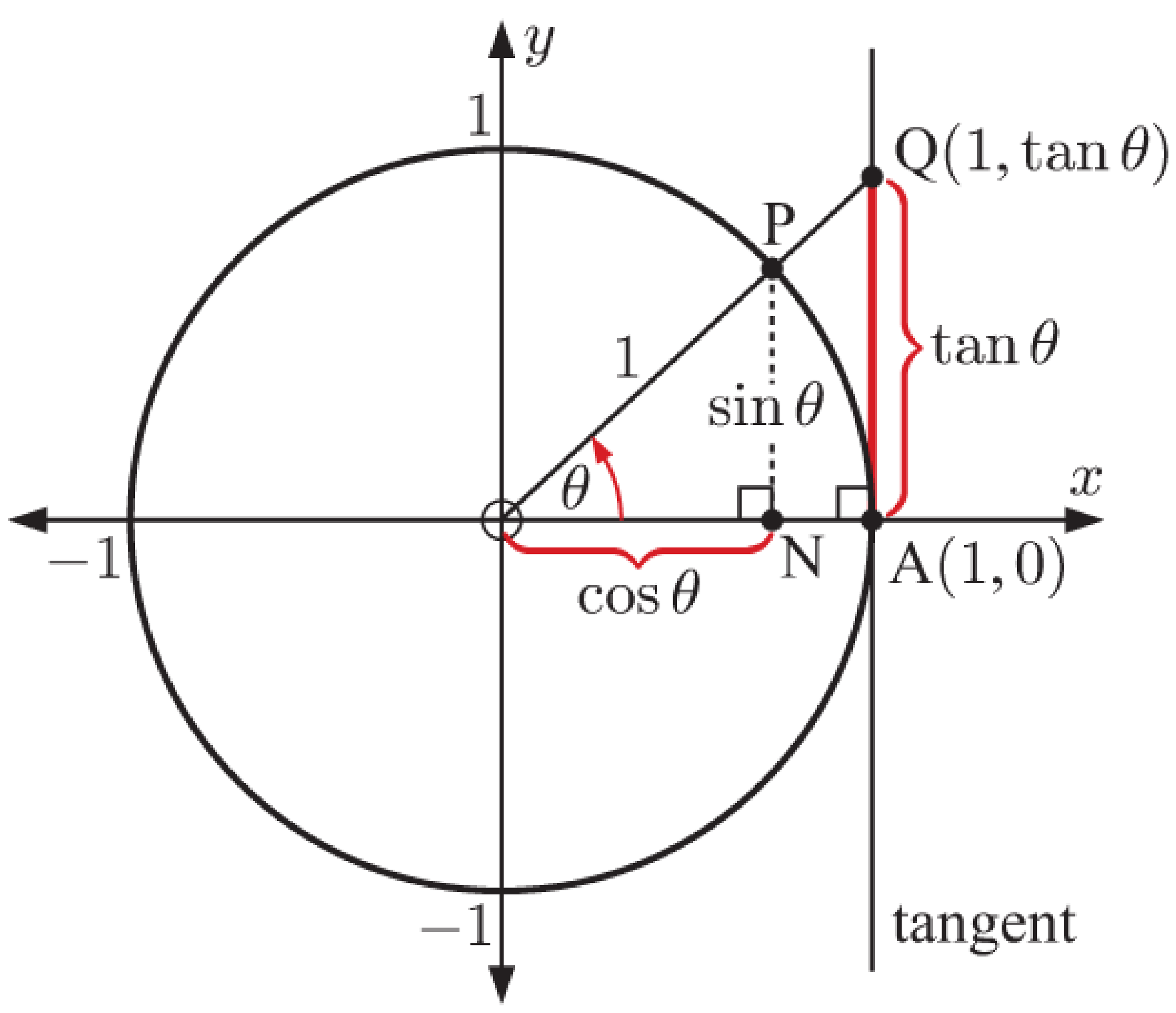
Suppose we extend [OP] to meet the tangent from $A(1, 0)$.

We let the intersection between these lines be point Q.

Note that as P moves, so does Q.

The position of Q relative to A is defined as the **tangent function**.

Notice that triangles ONP and OAQ are equiangular and therefore similar.



Consequently $\dfrac{\text{AQ}}{\text{OA}} = \dfrac{\text{NP}}{\text{ON}}$ and hence $\dfrac{\text{AQ}}{1} = \dfrac{\sin\theta}{\cos\theta}$.

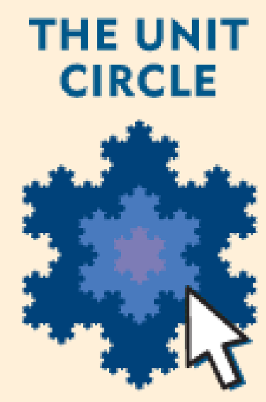
Under the definition that $\text{AQ} = \tan\theta$, $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

Since [OP] has gradient $\dfrac{\sin\theta}{\cos\theta}$, we can also say that $\tan\theta$ is the **gradient** of [OP].

INVESTIGATION — THE TRIGONOMETRIC RATIOS

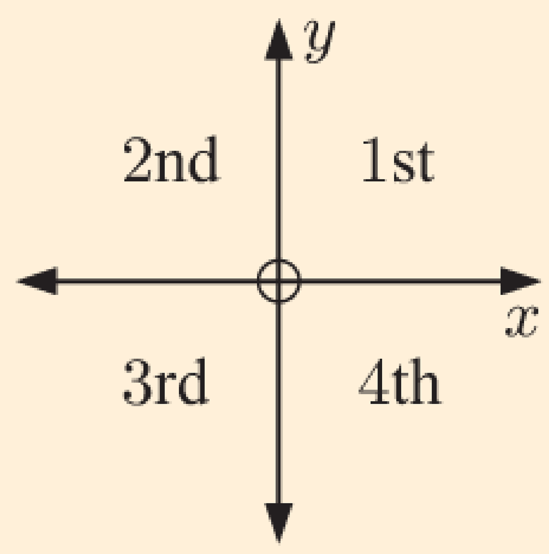
In this Investigation we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

What to do:



1 Click on the icon to run the Unit Circle software.
Drag the point P slowly around the circle.
Note the *sign* of each trigonometric ratio in each quadrant.

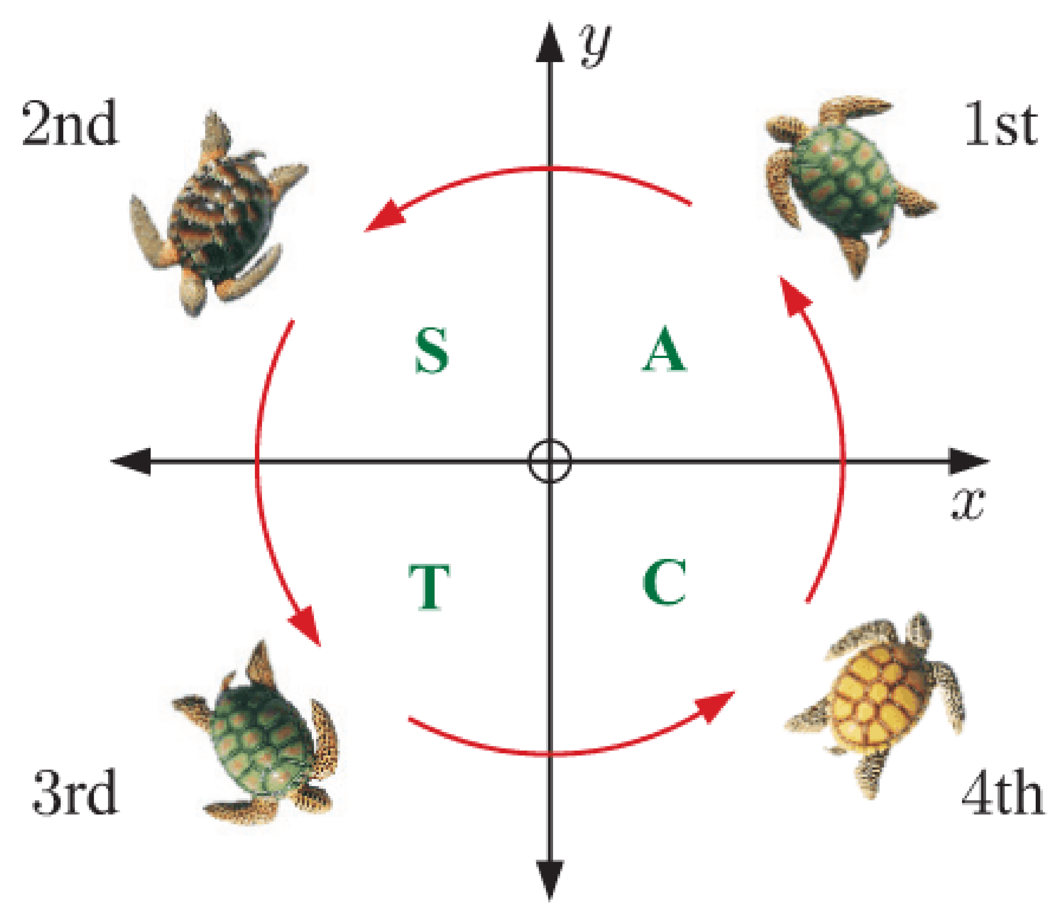
Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive		
2			
3			
4			



2 Hence write down the trigonometric ratios which are *positive* for each quadrant.

From the **Investigation** you should have discovered that:

- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are all positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.



We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.

PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ (in radians) then the position of P on the unit circle is unchanged.

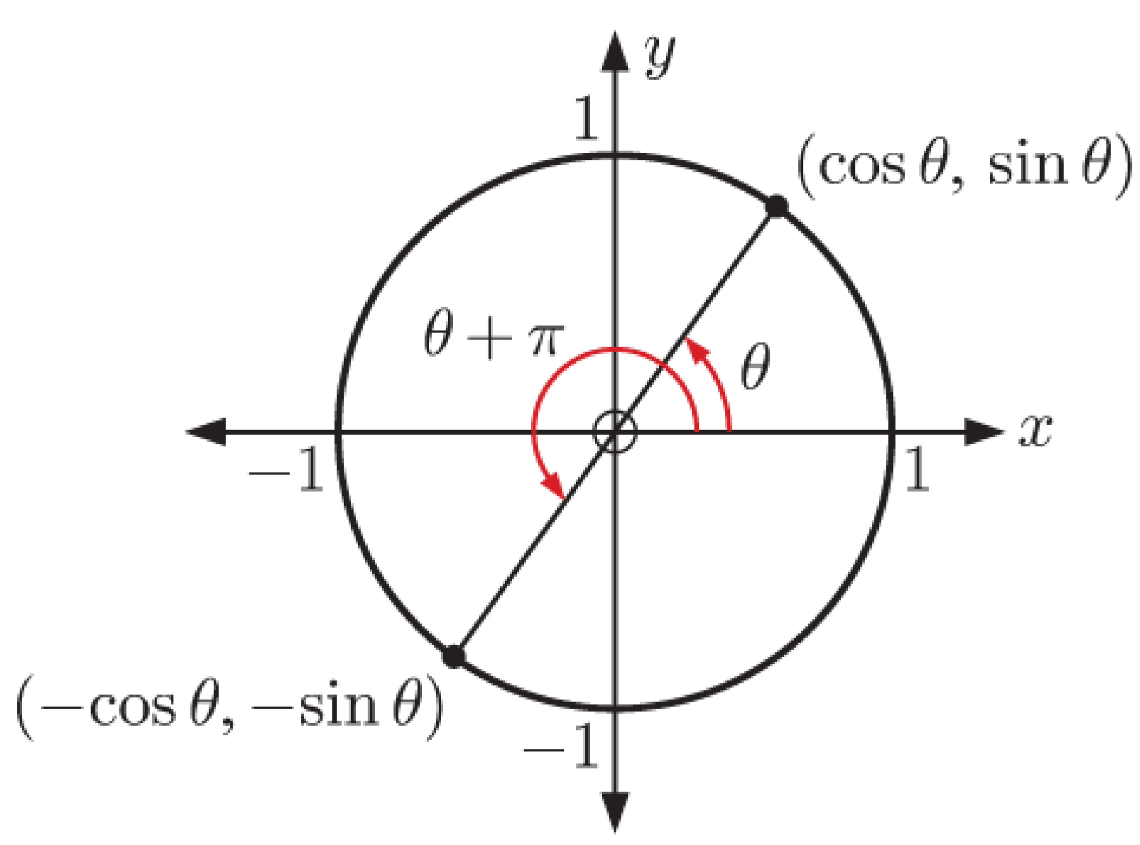
> For θ in radians and $k \in \mathbb{Z}$,
> $\mathbf{\cos(\theta + 2k\pi) = \cos\theta}$ and $\mathbf{\sin(\theta + 2k\pi) = \sin\theta}$.

We notice that for any point $(\cos\theta, \sin\theta)$ on the unit circle, the point directly opposite is $(-\cos\theta, -\sin\theta)$.

$$\therefore \quad \cos(\theta + \pi) = -\cos\theta$$

$$\sin(\theta + \pi) = -\sin\theta$$

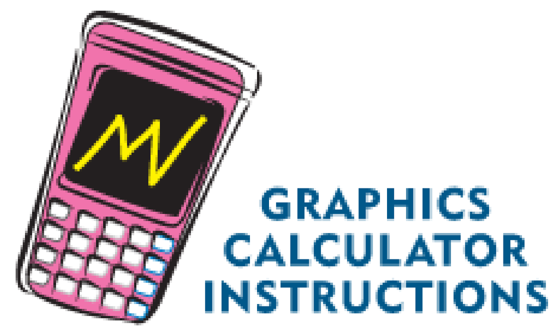
and $$\tan(\theta + \pi) = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$



> For θ in radians and $k \in \mathbb{Z}$, $\mathbf{\tan(\theta + k\pi) = \tan\theta}$.

CALCULATOR USE

When using your calculator to find trigonometric ratios for angles, you must make sure your calculator is correctly set to either **degree** or **radian** mode. Click on the icon for instructions.



EXERCISE 8C

1 With the aid of a unit circle, complete the following table:

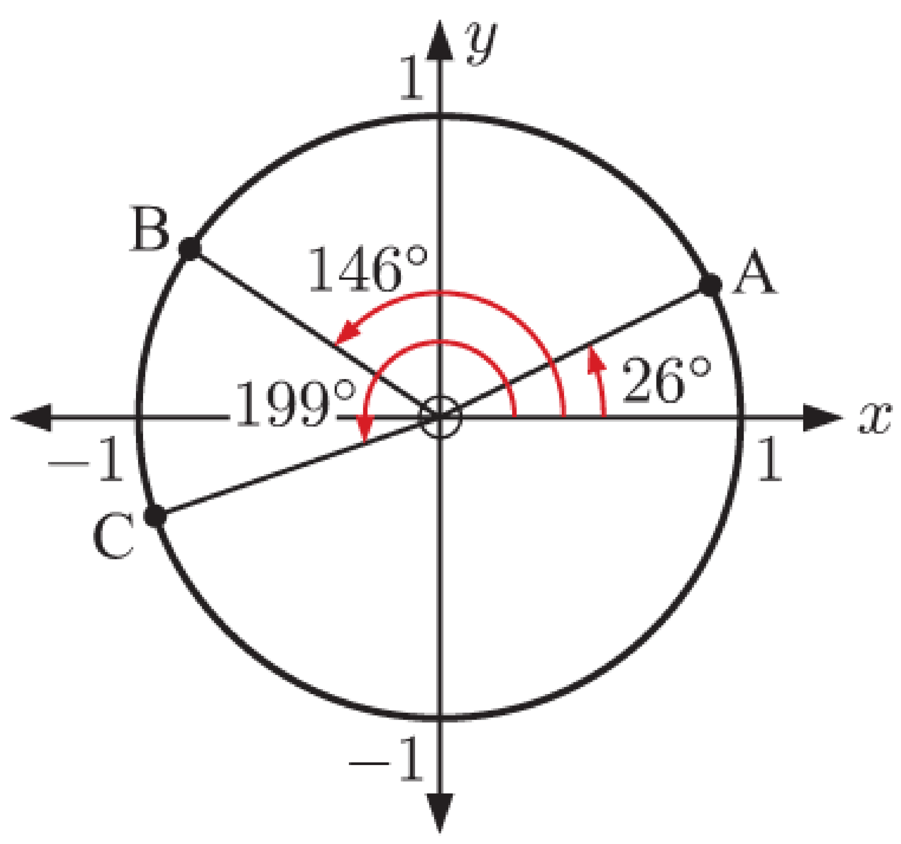
θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$
θ (radians)						
sine						
cosine						
tangent						

2 For each unit circle illustrated:

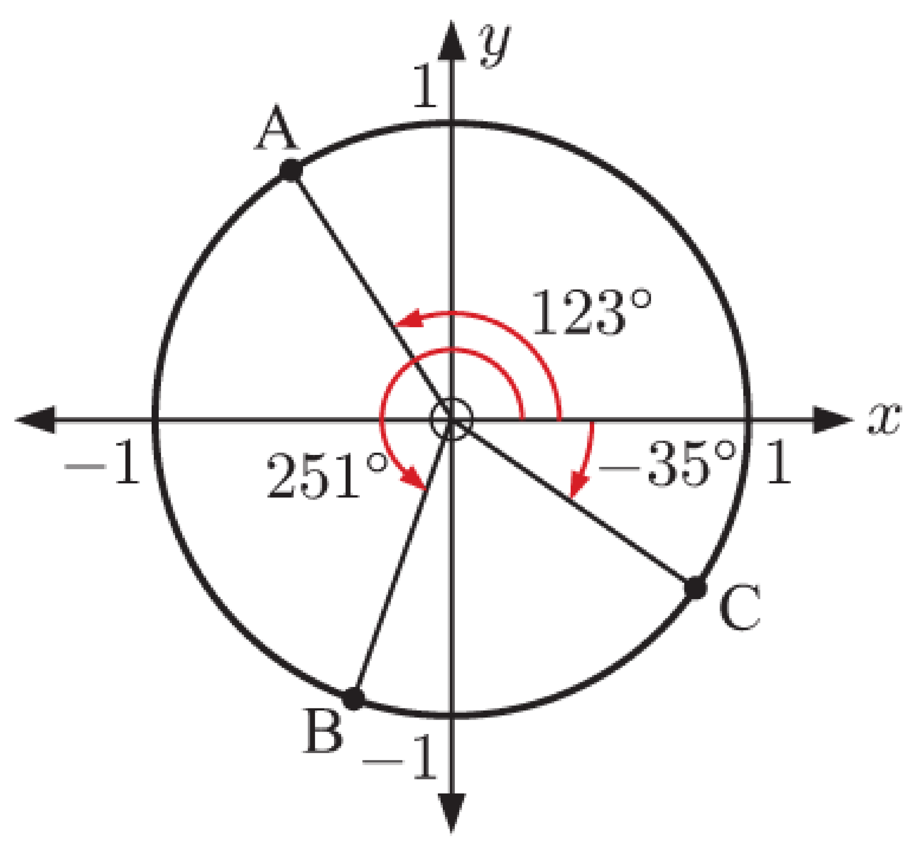
i State the exact coordinates of points A, B, and C in terms of sine and cosine.

ii Use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



3 **a** Use your calculator to evaluate:

i $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. Use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)							
sine							
cosine							
tangent							

4 Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

5 In which quadrants are the following true?

a $\cos\theta$ is positive.
b $\cos\theta$ is negative.
c $\cos\theta$ and $\sin\theta$ are both negative.
d $\cos\theta$ is negative and $\sin\theta$ is positive.

6 Explain why:

a $\cos 400° = \cos 40°$
b $\sin \frac{5\pi}{7} = \sin \frac{19\pi}{7}$
c $\tan \frac{13\pi}{8} = \tan\left(-\frac{11\pi}{8}\right)$

7 Which two of these have the same value?

A $\tan 15°$ **B** $\tan 50°$ **C** $\tan 200°$ **D** $\tan 230°$ **E** $\tan 300°$

8 Which two of these have the same value?

A $\sin 220°$ **B** $\sin \frac{2\pi}{9}$ **C** $\sin\left(-\frac{2\pi}{9}\right)$ **D** $\sin 120°$ **E** $\sin 40°$

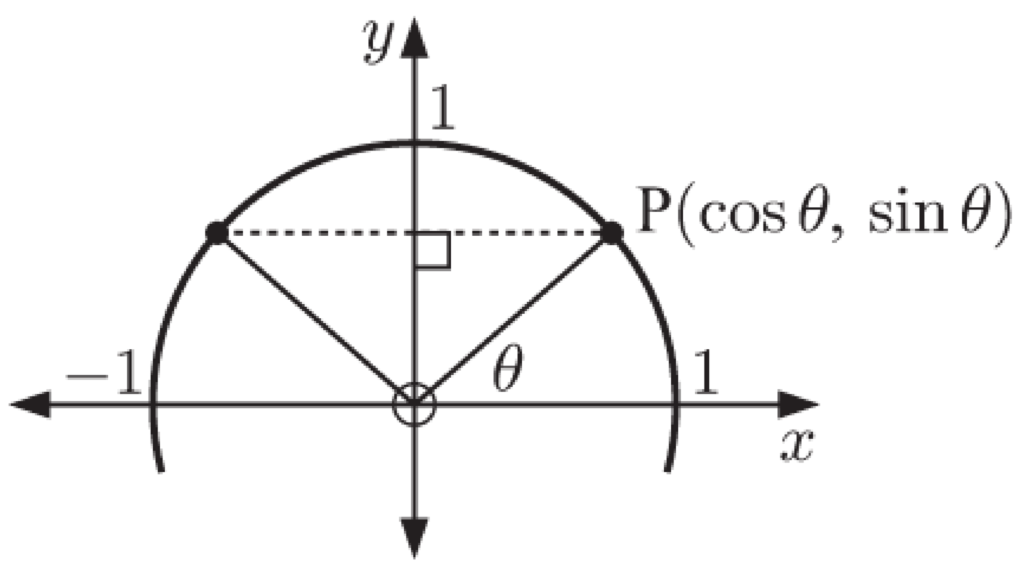
9 **a** Use your calculator to evaluate:

i $\sin 100°$ **ii** $\sin 80°$ **iii** $\sin 120°$ **iv** $\sin 60°$
v $\sin 150°$ **vi** $\sin 30°$ **vii** $\sin 45°$ **viii** $\sin 135°$

b Use the results from **a** to copy and complete: $\sin(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle with the same sine as:

i $45°$ **ii** $51°$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

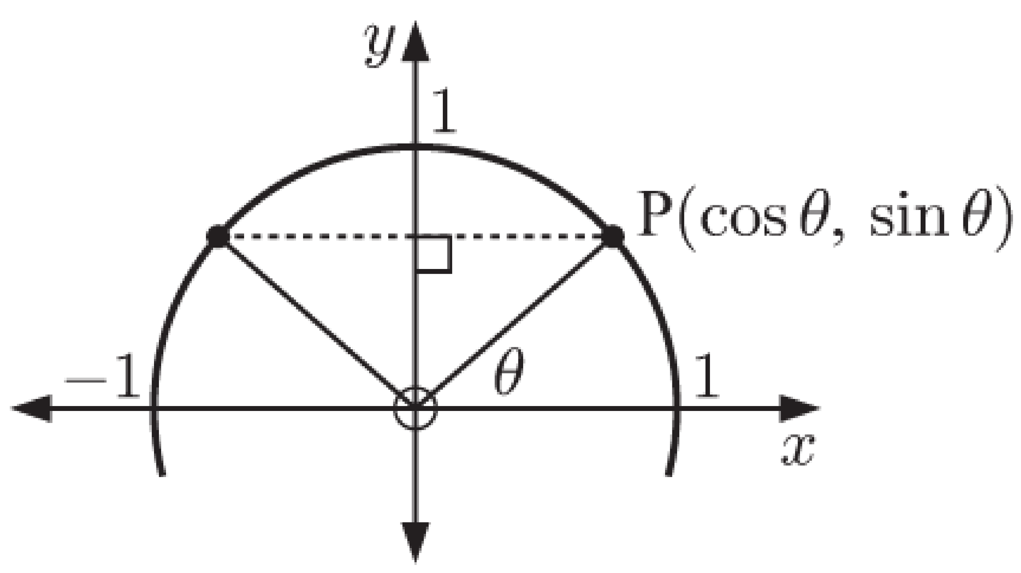
10 **a** Use your calculator to evaluate:

i $\cos 70°$ **ii** $\cos 110°$ **iii** $\cos 60°$ **iv** $\cos 120°$
v $\cos 25°$ **vi** $\cos 155°$ **vii** $\cos 80°$ **viii** $\cos 100°$

b Use the results from **a** to copy and complete: $\cos(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle which has the negative cosine of:

i $40°$ **ii** $19°$ **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

11 Use the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and your results from **9** and **10** to write $\tan(\pi - \theta)$ in terms of $\tan\theta$.

12 Without using your calculator, find:

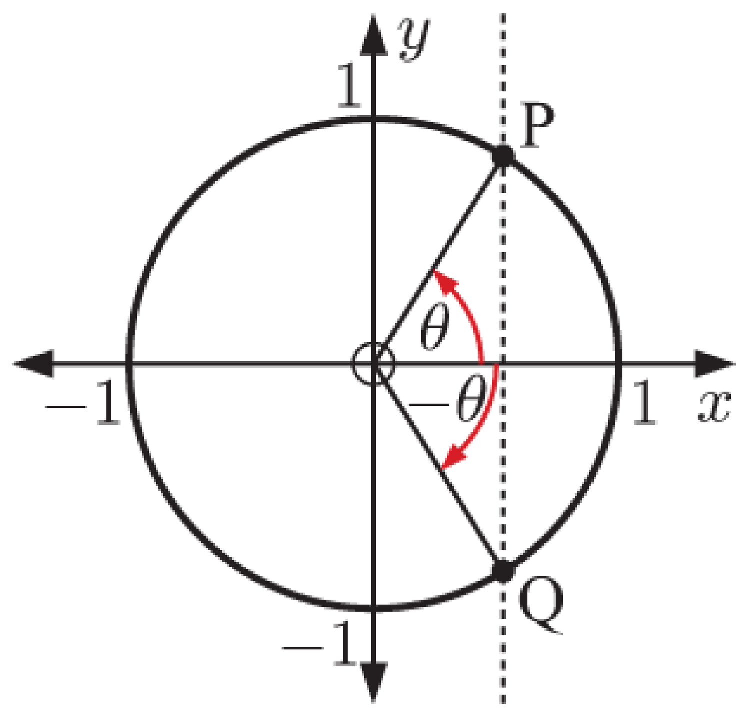
a $\sin 137°$ if $\sin 43° \approx 0.6820$
b $\sin 59°$ if $\sin 121° \approx 0.8572$
c $\cos 143°$ if $\cos 37° \approx 0.7986$
d $\cos 24°$ if $\cos 156° \approx -0.9135$
e $\sin 115°$ if $\sin 65° \approx 0.9063$
f $\cos 132°$ if $\cos 48° \approx 0.6691$

13 **a** Copy and complete:

θ (radians)	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
-1.17				

b What trigonometric formulae can be deduced from your results in **a**?

c The coordinates of P in the figure are $(\cos\theta, \sin\theta)$. By finding the coordinates of Q in terms of θ in *two different* ways, prove your formulae in **b**.

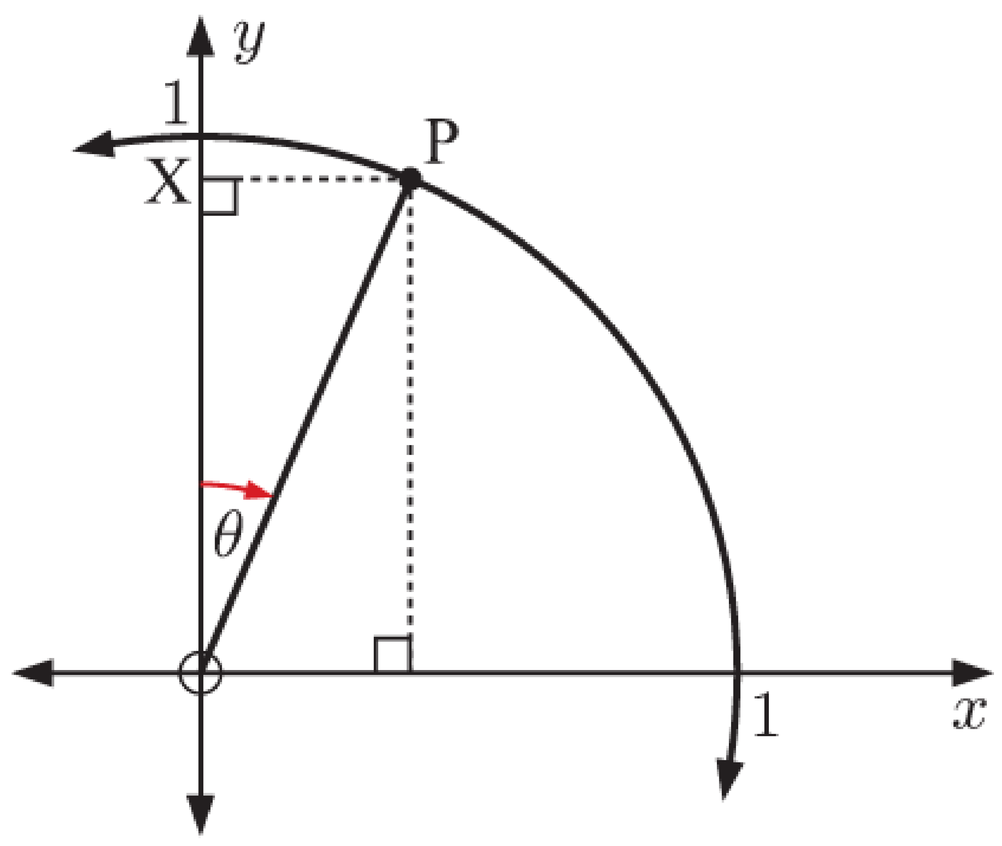


d Hence explain why

$$\cos(2\pi - \theta) = \cos\theta$$
and $$\sin(2\pi - \theta) = -\sin\theta.$$

e Write $\tan(2\pi - \theta)$ in terms of $\tan\theta$.

14 **a** Explain why P has coordinates $(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$.



b Show that:

i $XP = \sin\theta$ **ii** $OX = \cos\theta$

c Hence, copy and complete:

i $\cos(\frac{\pi}{2} - \theta) = ...$ **ii** $\sin(\frac{\pi}{2} - \theta) = ...$

d Check your answer to **c** by calculating:

i $\cos\frac{\pi}{5}$ and $\sin\frac{3\pi}{10}$ **ii** $\sin\frac{\pi}{8}$ and $\cos\frac{3\pi}{8}$.

e Write $\tan(\frac{\pi}{2} - \theta)$ in terms of $\tan\theta$.

DISCUSSION IDENTITIES

In the previous Exercise you should have proven the following trigonometric **identities**:

	Degree form	*Radian form*
Supplementary angles	$\cos(180° - \theta) = -\cos\theta$ $\sin(180° - \theta) = \sin\theta$ $\tan(180° - \theta) = -\tan\theta$	$\cos(\pi - \theta) = -\cos\theta$ $\sin(\pi - \theta) = \sin\theta$ $\tan(\pi - \theta) = -\tan\theta$
Negative angles	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$
Complementary angles	$\cos(90° - \theta) = \sin\theta$ $\sin(90° - \theta) = \cos\theta$ $\tan(90° - \theta) = \frac{1}{\tan\theta}$	$\cos(\frac{\pi}{2} - \theta) = \sin\theta$ $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ $\tan(\frac{\pi}{2} - \theta) = \frac{1}{\tan\theta}$

- What do we mean by the word "identity"?
- Why are identities important?
- What other identities do we use?

D MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

Angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ occur frequently in geometry, so it is important for us to write their trigonometric ratios exactly.

MULTIPLES OF $\frac{\pi}{4}$ OR $45°$

Consider $\theta = 45°$.

Angle OPB also measures $45°$, so triangle OBP is isosceles.

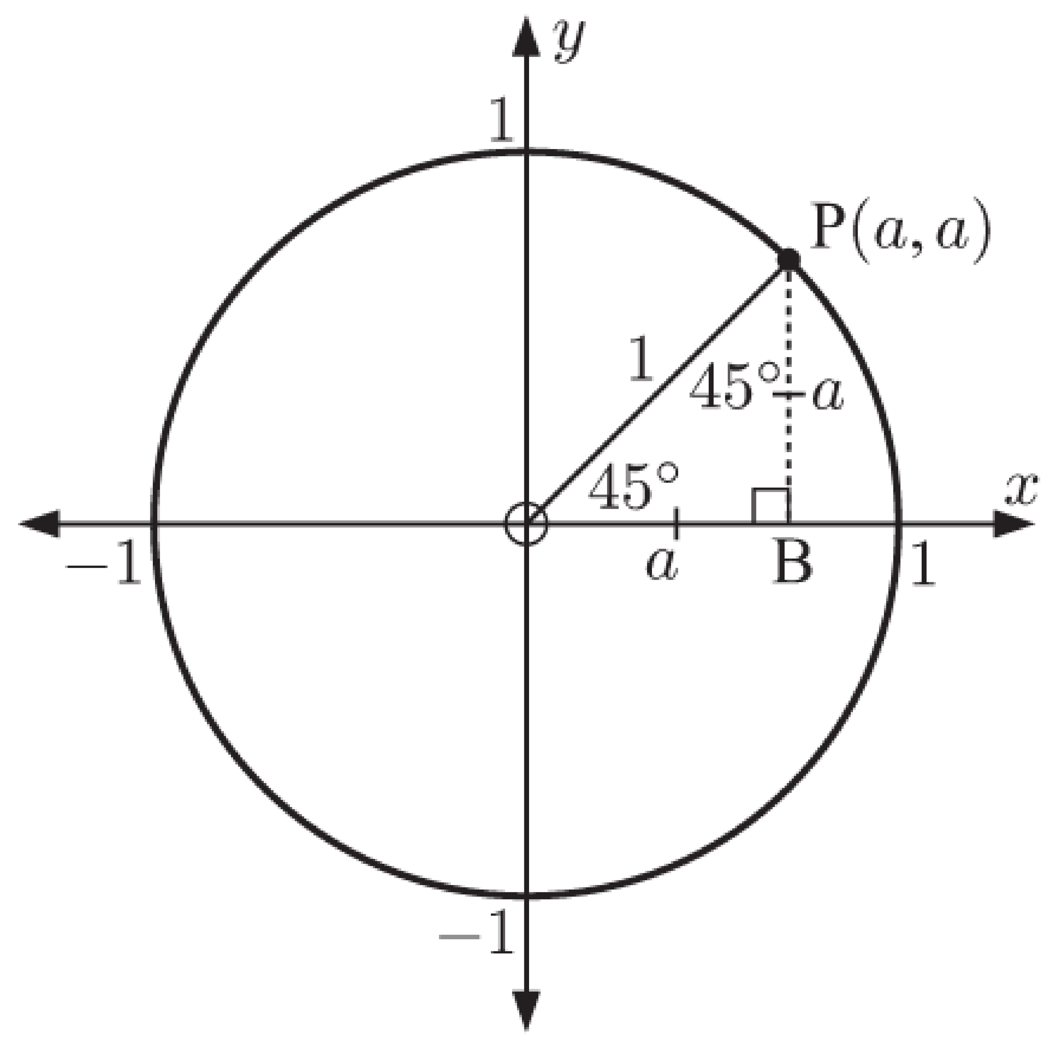
$\therefore$ we let $\text{OB} = \text{BP} = a$

Now $a^2 + a^2 = 1^2$ {Pythagoras}

$\therefore a^2 = \frac{1}{2}$

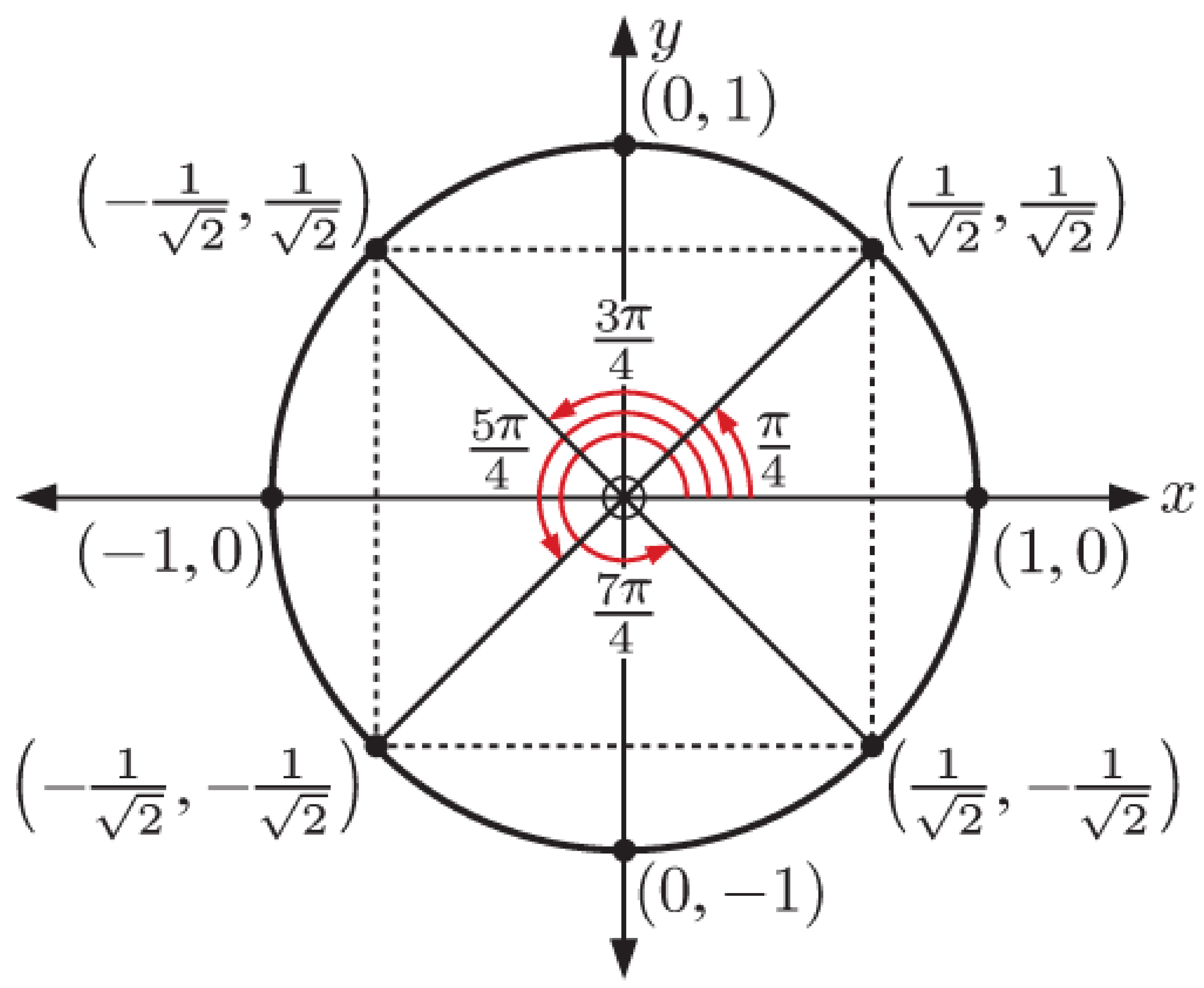
$\therefore a = \frac{1}{\sqrt{2}}$ {since $a > 0$}

$\therefore$ P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.707$.



So, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{4}$ by symmetry.



MULTIPLES OF $\frac{\pi}{6}$ OR $30°$

Consider $\theta = 60°$.

Since $\text{OA} = \text{OP}$, triangle OAP is isosceles.

Now $\widehat{\text{AOP}} = 60°$, so the remaining angles are therefore also $60°$. Triangle AOP is therefore equilateral.

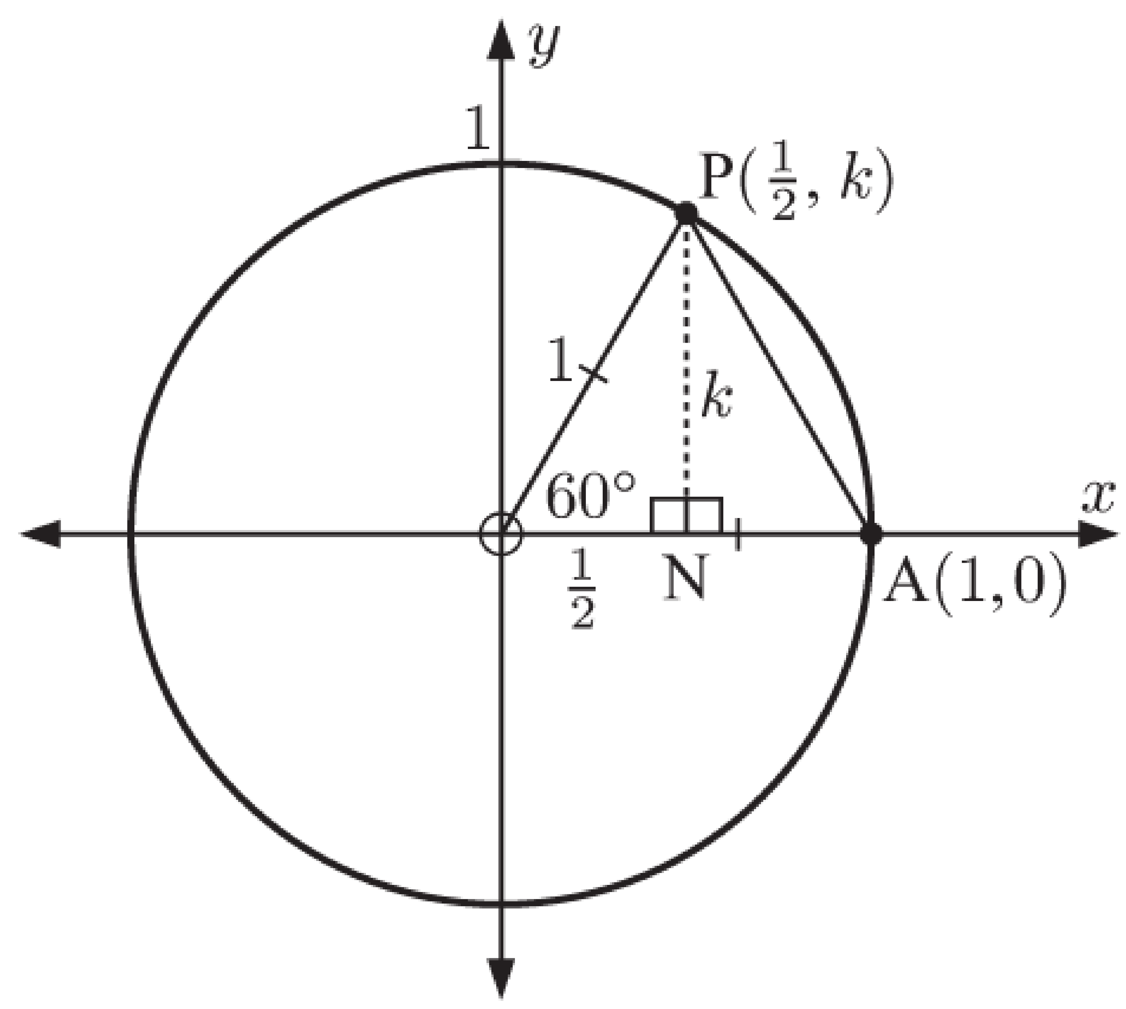
The altitude [PN] bisects base [OA], so $\text{ON} = \frac{1}{2}$.

If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

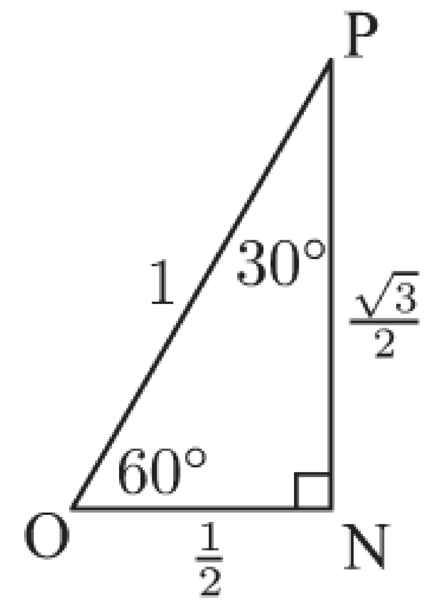
$\therefore k^2 = \frac{3}{4}$

$\therefore k = \frac{\sqrt{3}}{2}$ {since $k > 0$}

$\therefore$ P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $\frac{\sqrt{3}}{2} \approx 0.866$.



So, $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



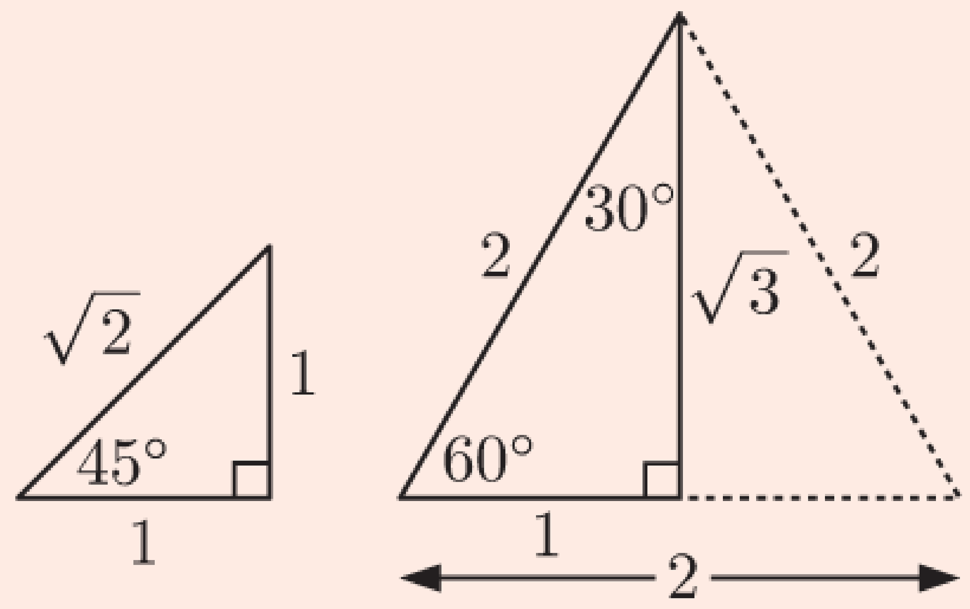
Now $\widehat{\text{NPO}} = \frac{\pi}{6} = 30°$. Hence $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

DISCUSSION

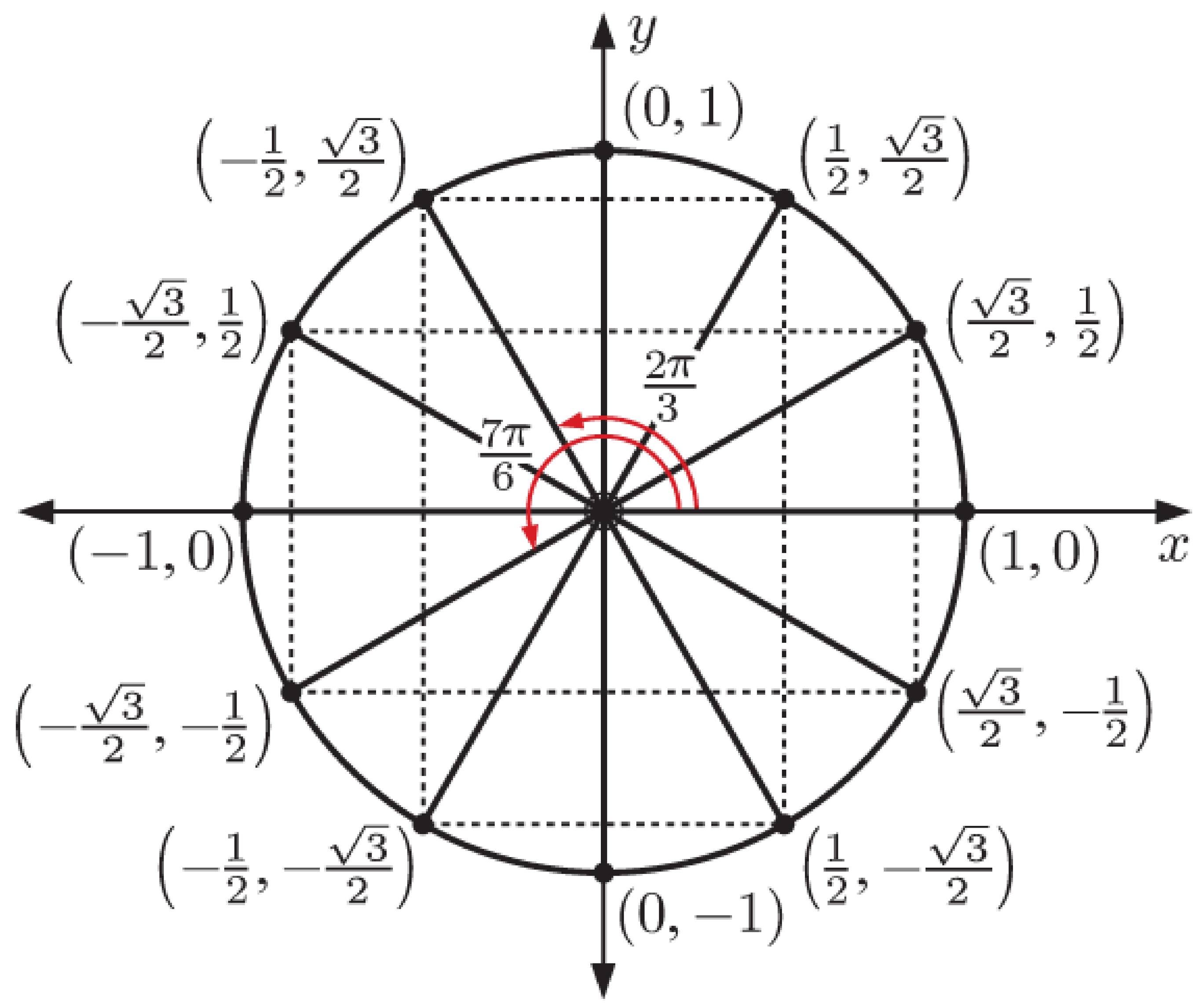
You should remember the values of cosine and sine for angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

However, if you forget, you can use these diagrams to quickly generate the results.

Discuss how you can do this.



We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{6}$ by symmetry.



SUMMARY

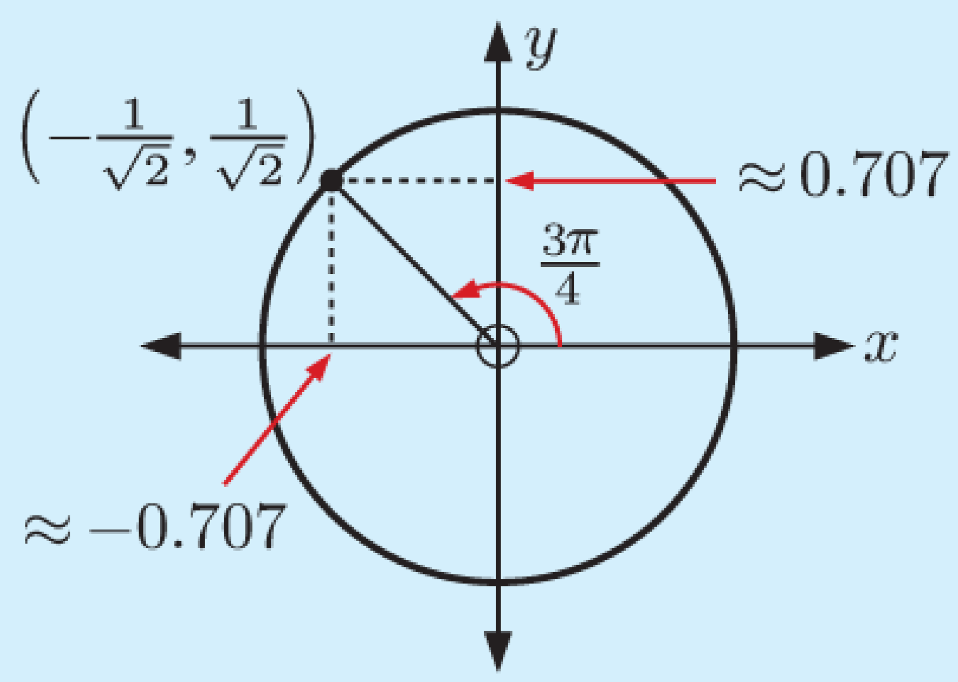
- For **multiples of $\frac{\pi}{2}$**, the coordinates of the points on the unit circle involve 0 and ± 1.
- For *other* **multiples of $\frac{\pi}{4}$**, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- For *other* **multiples of $\frac{\pi}{6}$**, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

Example 5 — Self Tutor

Find the exact values of $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for:

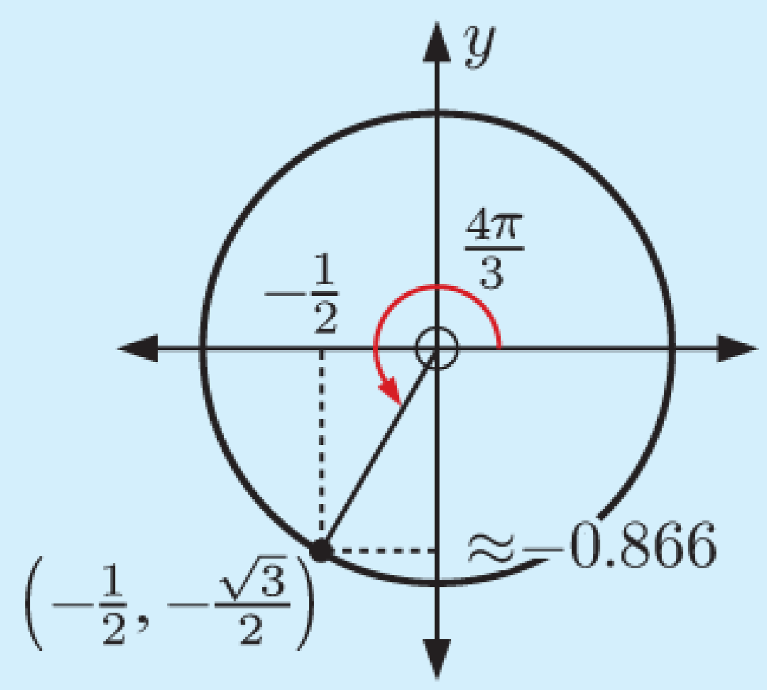
a $\alpha = \frac{3\pi}{4}$ **b** $\alpha = \frac{4\pi}{3}$

a



$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$
$$\tan\frac{3\pi}{4} = -1$$

b



$$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$
$$\cos\frac{4\pi}{3} = -\frac{1}{2}$$
$$\tan\frac{4\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

EXERCISE 8D

1 Use a unit circle diagram to find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{3\pi}{4}$ **c** $\frac{7\pi}{4}$ **d** π **e** $-\frac{3\pi}{4}$

2 Use a unit circle diagram to find exact values for $\sin\beta$, $\cos\beta$, and $\tan\beta$ for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

3 Find the exact values of:

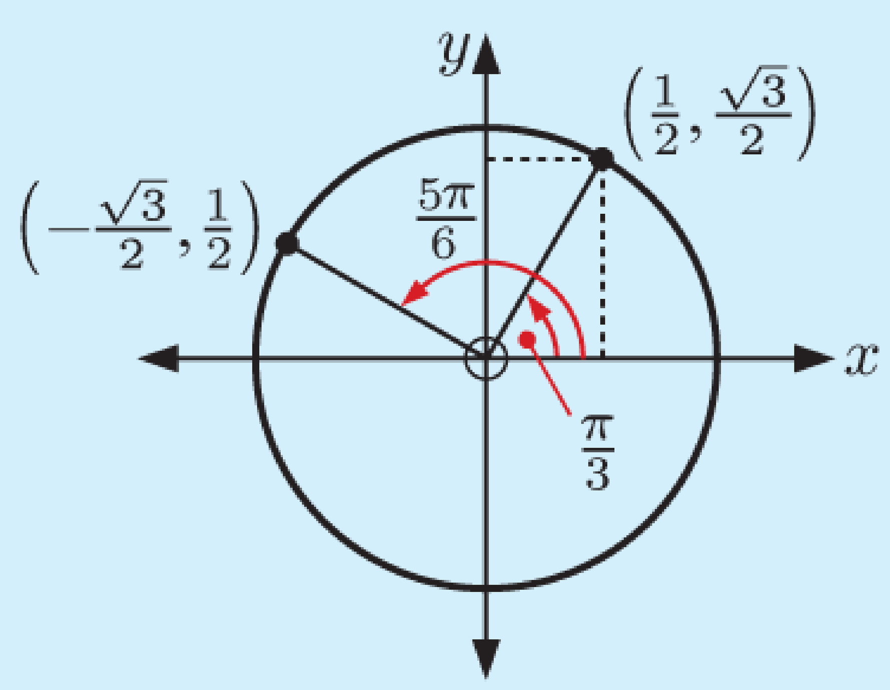
a $\cos\frac{2\pi}{3}$, $\sin\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ **b** $\cos\left(-\frac{\pi}{4}\right)$, $\sin\left(-\frac{\pi}{4}\right)$, and $\tan\left(-\frac{\pi}{4}\right)$

4 **a** Find the exact values of $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

b What can you say about $\tan\frac{\pi}{2}$?

Example 6 **Self Tutor**

Without using a calculator, show that $8\sin\frac{\pi}{3}\cos\frac{5\pi}{6} = -6$.



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$$\therefore\ 8\sin\frac{\pi}{3}\cos\frac{5\pi}{6} = 8\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= 2(-3)$$
$$= -6$$

5 Without using a calculator, evaluate:

a $\sin^2\left(\frac{\pi}{3}\right)$ **b** $\sin\frac{\pi}{6}\cos\frac{\pi}{3}$ **c** $1-\cos^2\left(\frac{\pi}{6}\right)$

d $\sin^2\left(\frac{2\pi}{3}\right)-1$ **e** $\cos^2\left(\frac{\pi}{4}\right)-\sin\frac{7\pi}{6}$ **f** $\sin\frac{3\pi}{4}-\cos\frac{5\pi}{4}$

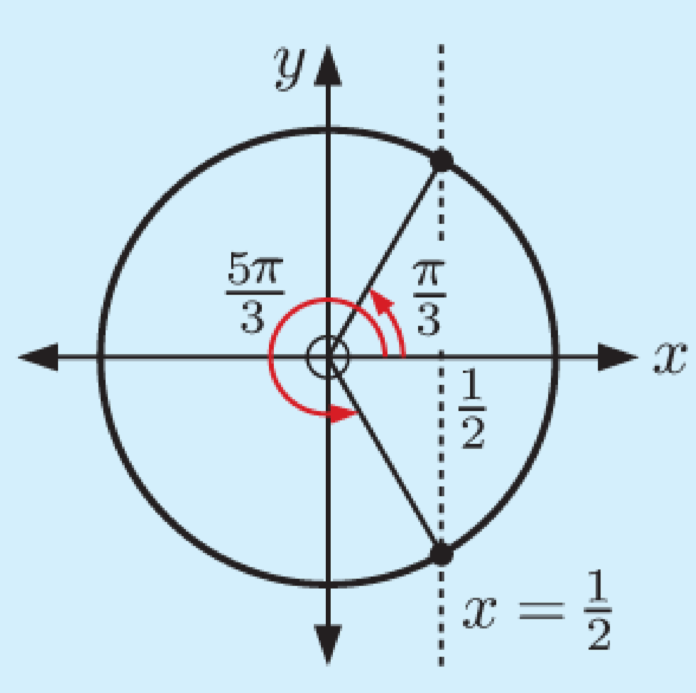
g $1-2\sin^2\left(\frac{7\pi}{6}\right)$ **h** $\cos^2\left(\frac{5\pi}{6}\right)-\sin^2\left(\frac{5\pi}{6}\right)$ **i** $\tan^2\left(\frac{\pi}{3}\right)-2\sin^2\left(\frac{\pi}{4}\right)$

j $2\tan\left(-\frac{5\pi}{4}\right)-\sin\frac{3\pi}{2}$ **k** $\dfrac{2\tan\frac{5\pi}{6}}{1-\tan^2\left(\frac{5\pi}{6}\right)}$ **l** $\dfrac{\cos\frac{\pi}{3}}{\sin\frac{4\pi}{3}+\tan\frac{\pi}{6}}$

Check all answers using your calculator.

Example 7 **Self Tutor**

Find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



Since the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

6 Find all angles between 0 and 2π with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

7 Find all angles between 0 and 2π (inclusive) which have a tangent of:

a 1 **b** -1 **c** $\sqrt{3}$ **d** 0 **e** $\frac{1}{\sqrt{3}}$ **f** $-\sqrt{3}$

8 Find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

9 Find θ if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

i $\tan\theta = -\frac{1}{\sqrt{3}}$ **j** $\tan^2\theta = 3$

10 Find *all* values of θ for which $\tan\theta$ is:

a zero **b** undefined.

E THE PYTHAGOREAN IDENTITY

From the equation of the unit circle $x^2 + y^2 = 1$, we obtain the **Pythagorean identity**:

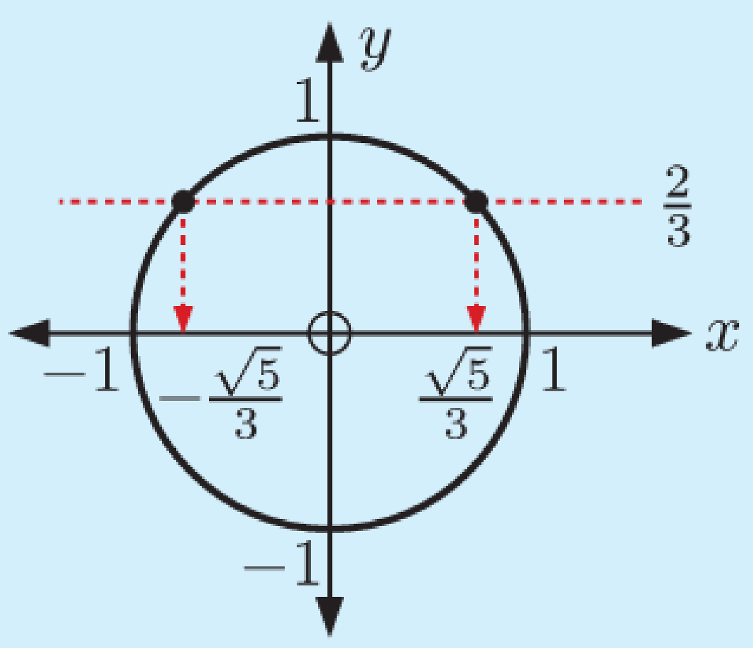
For any angle θ, $\cos^2\theta + \sin^2\theta = 1$.

We can use this identity to find one trigonometric ratio from another.

Example 8 **Self Tutor**

Find the possible exact values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos^2\theta + \left(\frac{2}{3}\right)^2 = 1$$
$$\therefore \cos^2\theta = \frac{5}{9}$$
$$\therefore \cos\theta = \pm\frac{\sqrt{5}}{3}$$



EXERCISE 8E

1 Find the possible exact values of $\cos\theta$ for:

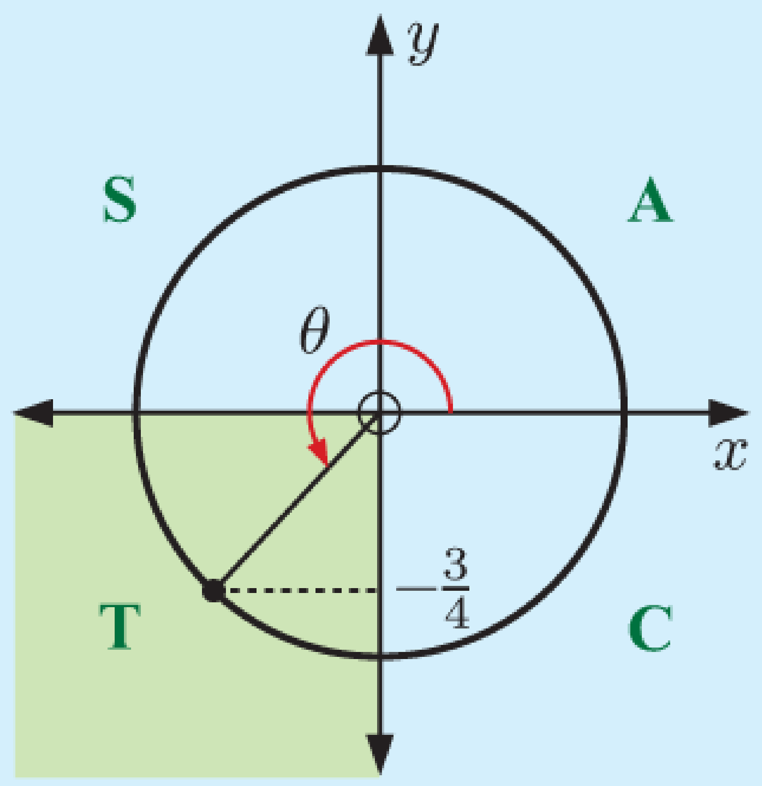
a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{3}$ **c** $\sin\theta = 0$ **d** $\sin\theta = -1$

2 Find the possible exact values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

Example 9 **Self Tutor**

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give exact values.



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore \cos^2\theta + \frac{9}{16} = 1$

$\therefore \cos^2\theta = \frac{7}{16}$

$\therefore \cos\theta = \pm\frac{\sqrt{7}}{4}$

But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle.

$\therefore$ $\cos\theta$ is negative.

$\therefore$ $\cos\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}}$

3 Without using a calculator, find:

a $\sin\theta$ if $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\cos\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$

c $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

d $\sin\theta$ if $\cos\theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Find $\tan\theta$ exactly given:

a $\sin\theta = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

b $\cos\theta = \frac{1}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

c $\sin\theta = -\frac{1}{\sqrt{3}}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\cos\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Example 10 **Self Tutor**

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give exact answers.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = -2$

$\therefore \sin\theta = -2\cos\theta$

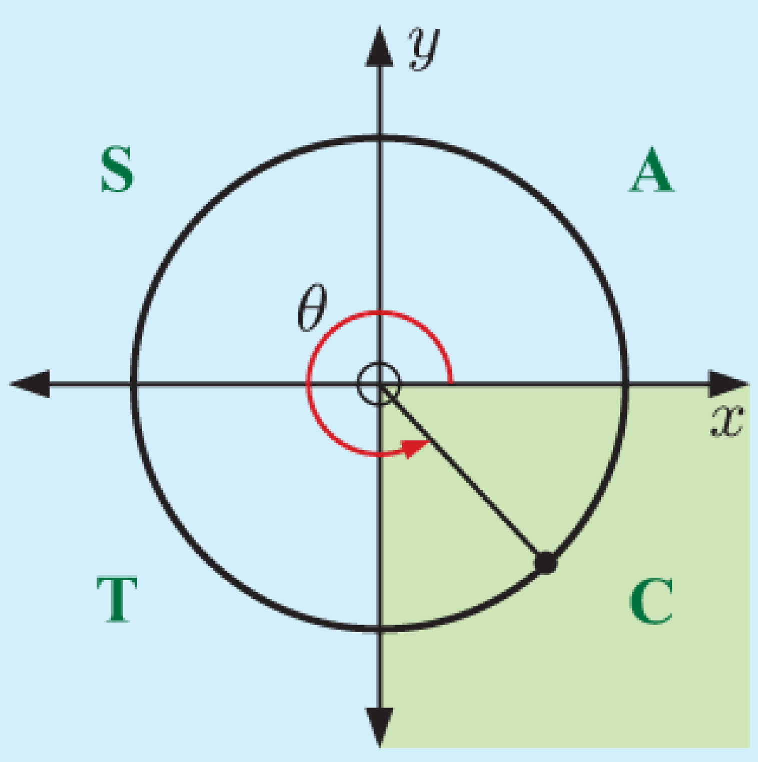
Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore (-2\cos\theta)^2 + \cos^2\theta = 1$

$\therefore 4\cos^2\theta + \cos^2\theta = 1$

$\therefore 5\cos^2\theta = 1$

$\therefore \cos\theta = \pm\frac{1}{\sqrt{5}}$



But $\frac{3\pi}{2} < \theta < 2\pi$, so θ is a quadrant 4 angle. $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

5 Find exact values for $\sin\theta$ and $\cos\theta$ given that:

a $\tan\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$

c $\tan\theta = \frac{\sqrt{5}}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\tan\theta = -\frac{12}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

6 Suppose $\tan\theta = k$ where k is a constant and $\pi < \theta < \frac{3\pi}{2}$. Write expressions for $\sin\theta$ and $\cos\theta$ in terms of k.

F FINDING ANGLES

In **Exercise 8C** you should have proven that:

For θ in degrees:

- $\sin(180° - \theta) = \sin\theta$
- $\cos(180° - \theta) = -\cos\theta$
- $\cos(360° - \theta) = \cos\theta$
- $\sin(360° - \theta) = -\sin\theta$

For θ in radians:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\sin(2\pi - \theta) = -\sin\theta$

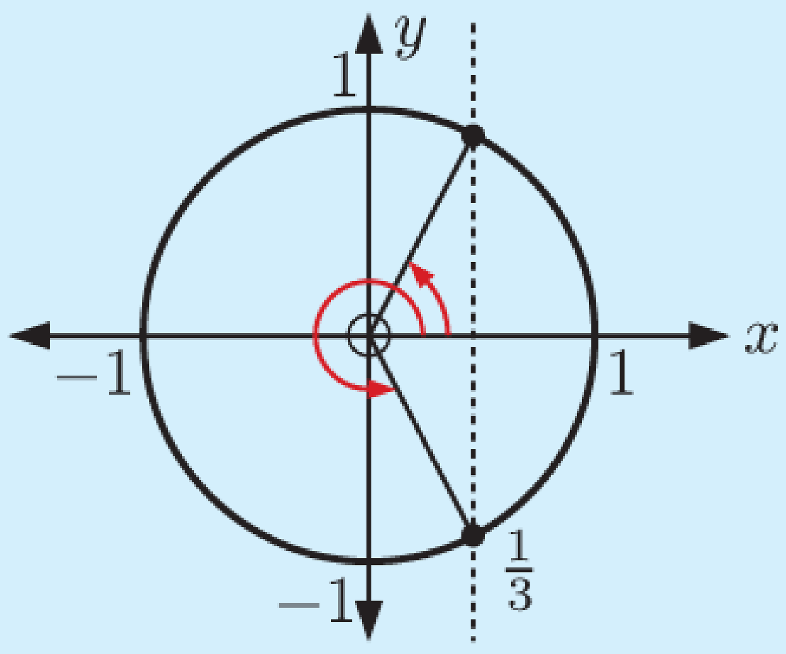
We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

Example 11 Self Tutor

Find the two angles θ on the unit circle, with $0° \leqslant \theta \leqslant 360°$, such that:

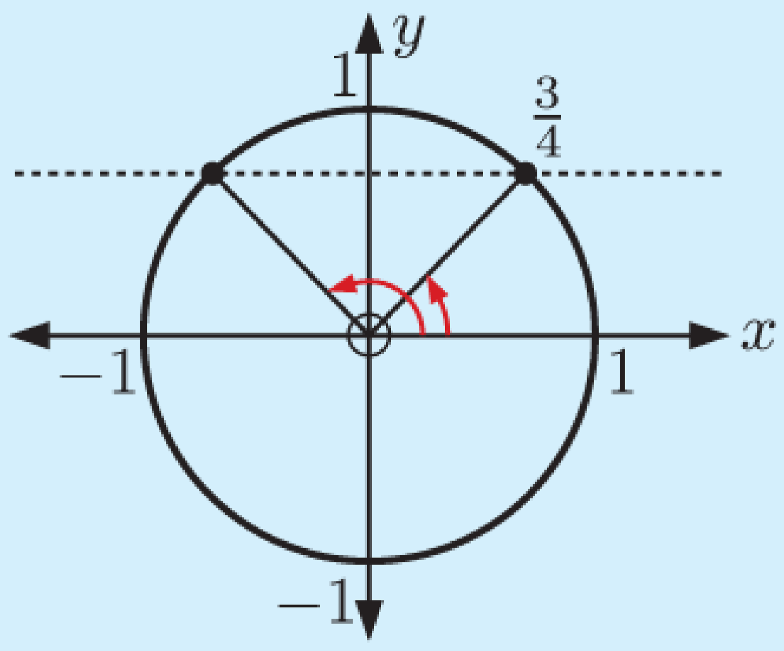
a $\cos\theta = \frac{1}{3}$ **b** $\sin\theta = \frac{3}{4}$ **c** $\tan\theta = 2$

a Using technology, $\cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°$



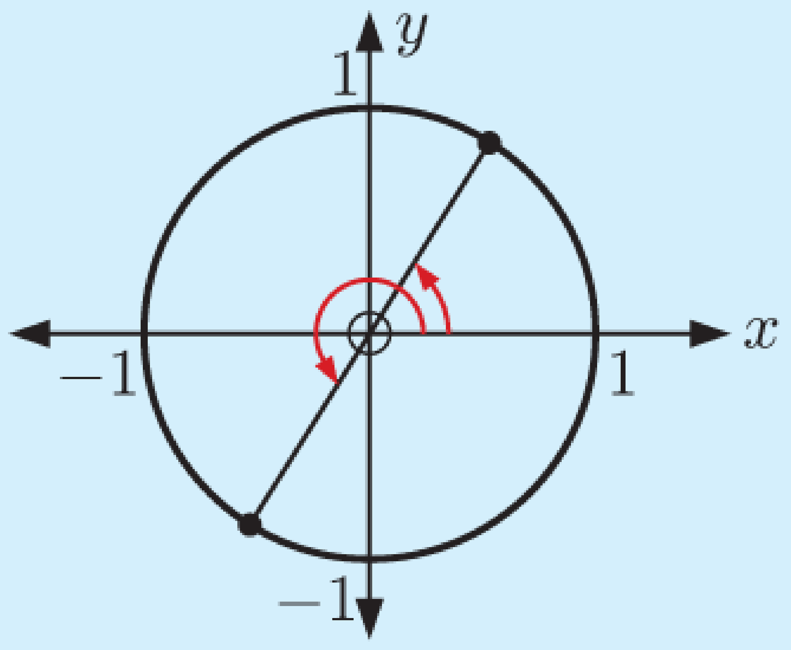
$\therefore \theta \approx 70.53°$ or $360° - 70.53°$
$\therefore \theta \approx 70.5°$ or $289.5°$

b Using technology, $\sin^{-1}\left(\frac{3}{4}\right) \approx 48.59°$

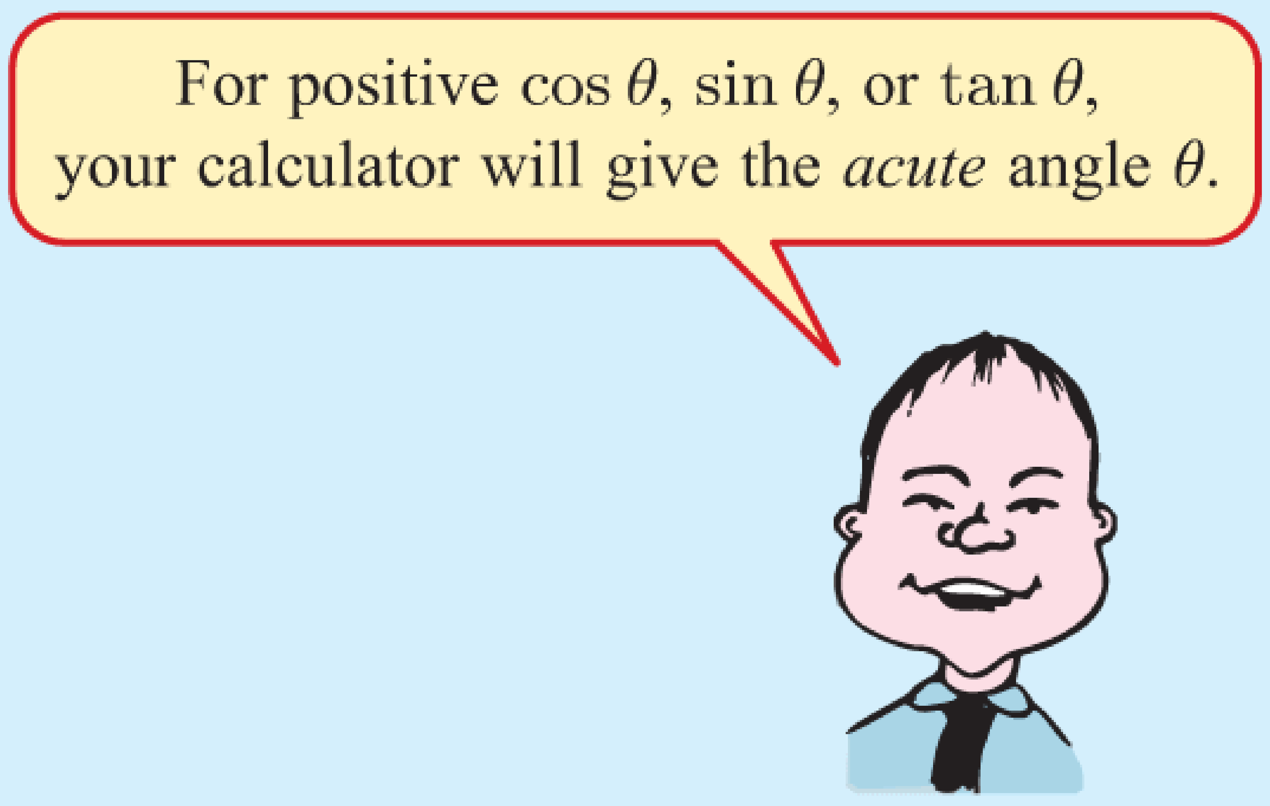


$\therefore \theta \approx 48.59°$ or $180° - 48.59°$
$\therefore \theta \approx 48.6°$ or $131.4°$

c Using technology, $\tan^{-1}(2) \approx 63.43°$



$\therefore \theta \approx 63.43°$ or $180° + 63.43°$
$\therefore \theta \approx 63.4°$ or $243.4°$



EXERCISE 8F

1 Find two angles θ on the unit circle, with $0° \leqslant \theta \leqslant 360°$, such that:

a $\tan\theta = 4$ **b** $\cos\theta = 0.83$ **c** $\sin\theta = \frac{3}{5}$

d $\cos\theta = 0$ **e** $\tan\theta = 6.67$ **f** $\cos\theta = \frac{2}{17}$

2 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\tan\theta = \frac{1}{3}$ **b** $\cos\theta = \frac{3}{7}$ **c** $\sin\theta = 0.61$

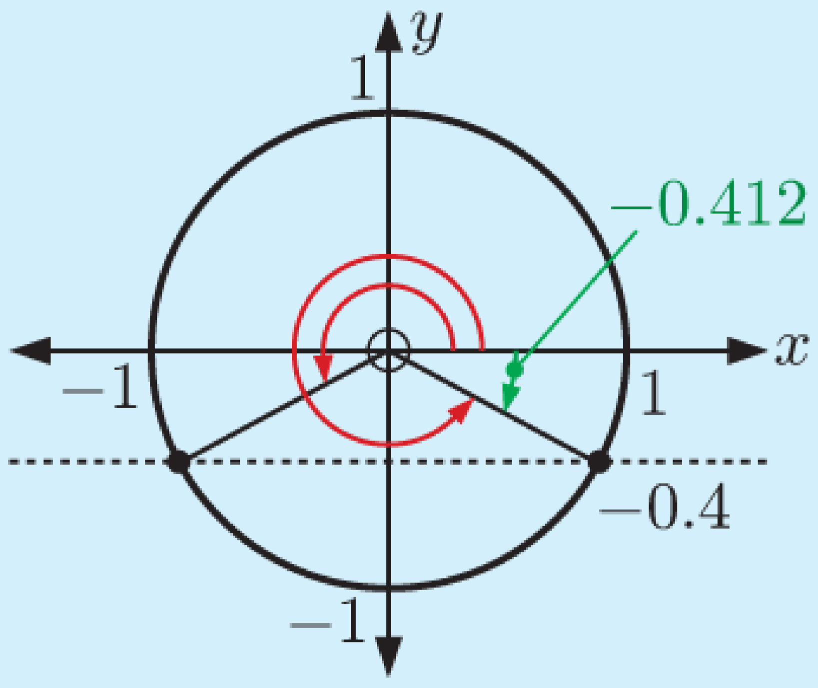
d $\cos\theta = \frac{1}{4}$ **e** $\tan\theta = 0.114$ **f** $\sin\theta = \frac{1}{6}$

Example 12 Self Tutor

Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\sin\theta = -0.4$ **b** $\cos\theta = -\frac{2}{3}$ **c** $\tan\theta = -\frac{1}{3}$

a Using technology, $\sin^{-1}(-0.4) \approx -0.412$



If $\sin\theta$ or $\tan\theta$ is negative, your calculator will give θ in the domain $-\frac{\pi}{2} < \theta < 0$.

If $\cos\theta$ is negative, your calculator will give the *obtuse* angle θ.

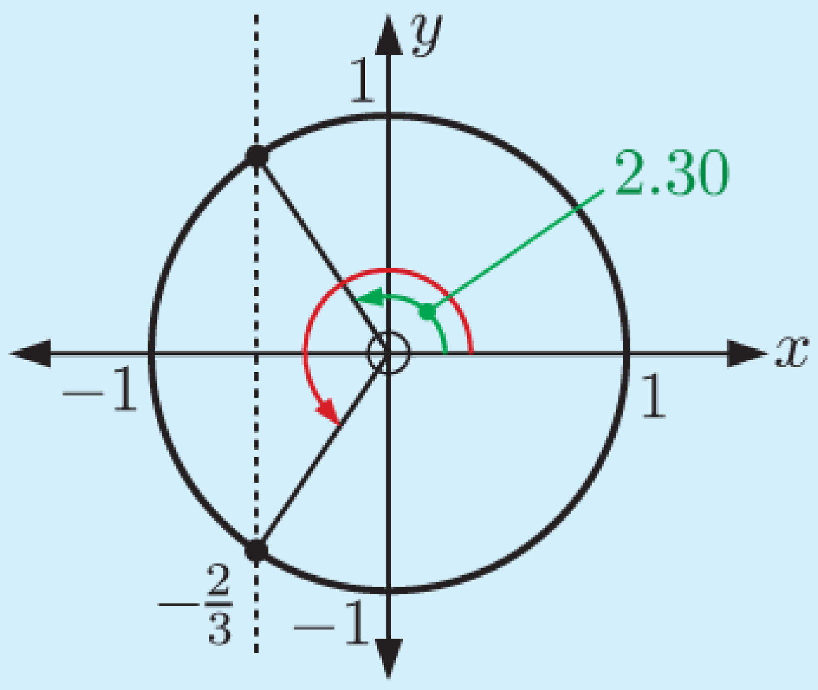
The angles given by your calculator are shown in green.

But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \theta \approx \pi + 0.412$ or $2\pi - 0.412$

$\therefore \theta \approx 3.55$ or 5.87

b Using technology, $\cos^{-1}\left(-\frac{2}{3}\right) \approx 2.30$

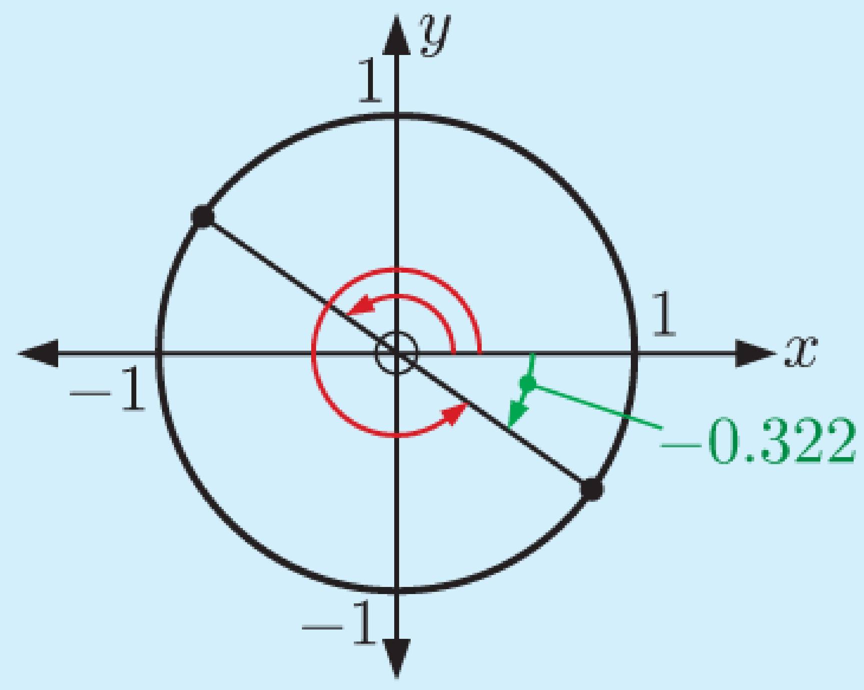


But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \theta \approx 2.30$ or $2\pi - 2.30$

$\therefore \theta \approx 2.30$ or 3.98

c Using technology, $\tan^{-1}\left(-\frac{1}{3}\right) \approx -0.322$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \theta \approx \pi - 0.322$ or $2\pi - 0.322$

$\therefore \theta \approx 2.82$ or 5.96

3 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = -\frac{1}{4}$ **b** $\sin\theta = 0$ **c** $\tan\theta = -3.1$ **d** $\sin\theta = -0.421$

e $\tan\theta = 1.2$ **f** $\cos\theta = 0.7816$ **g** $\sin\theta = \frac{1}{11}$ **h** $\cos\theta = -\frac{1}{\sqrt{3}}$

4 Find all θ such that $-180° \leqslant \theta \leqslant 180°$ and:

a $\cos\theta = -\frac{1}{10}$ **b** $\sin\theta = \frac{4}{5}$ **c** $\tan\theta = -\frac{3}{2}$

d $\cos\theta = 0.8$ **e** $\tan\theta = -\frac{5}{6}$ **f** $\sin\theta = -\frac{7}{11}$

5 **a** Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that $\cos\theta = \frac{3}{10}$.

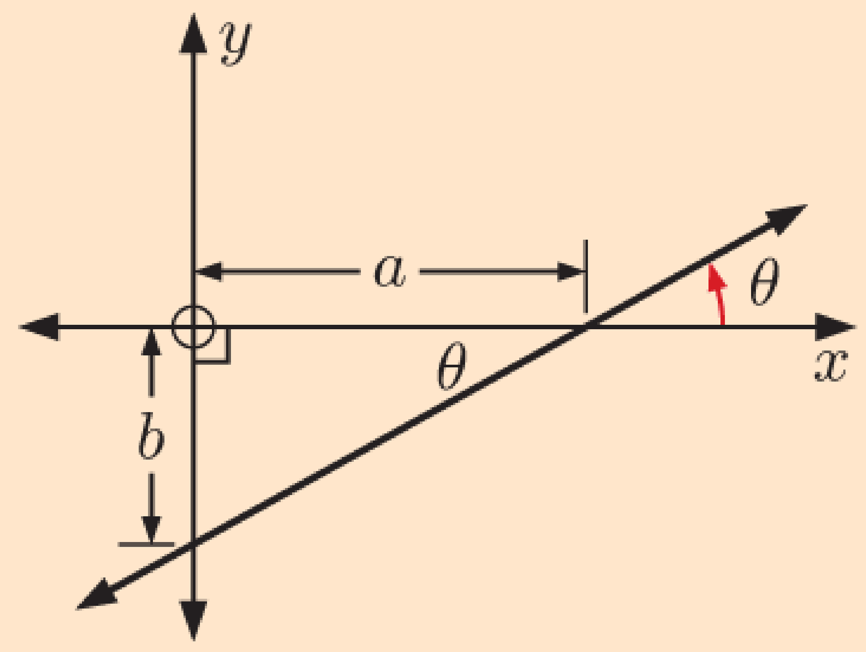
b For each value of θ, find $\sin\theta$ and $\tan\theta$ exactly.

G THE EQUATION OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its gradient is $m = \tan\theta$.

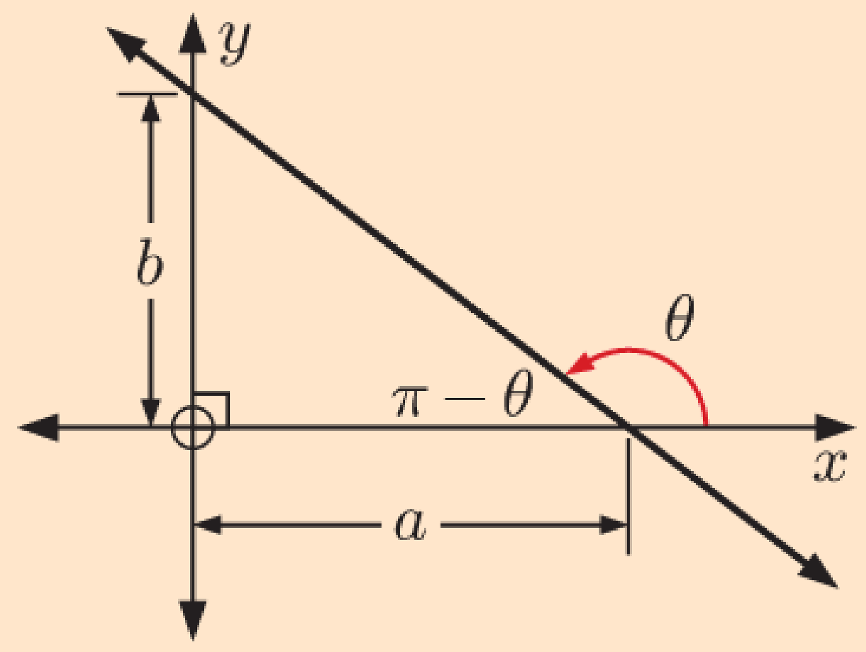
Proof:

- For $m \geqslant 0$:



$$\text{Gradient } m = \frac{0 - -b}{a - 0} = \frac{b}{a} = \tan\theta$$

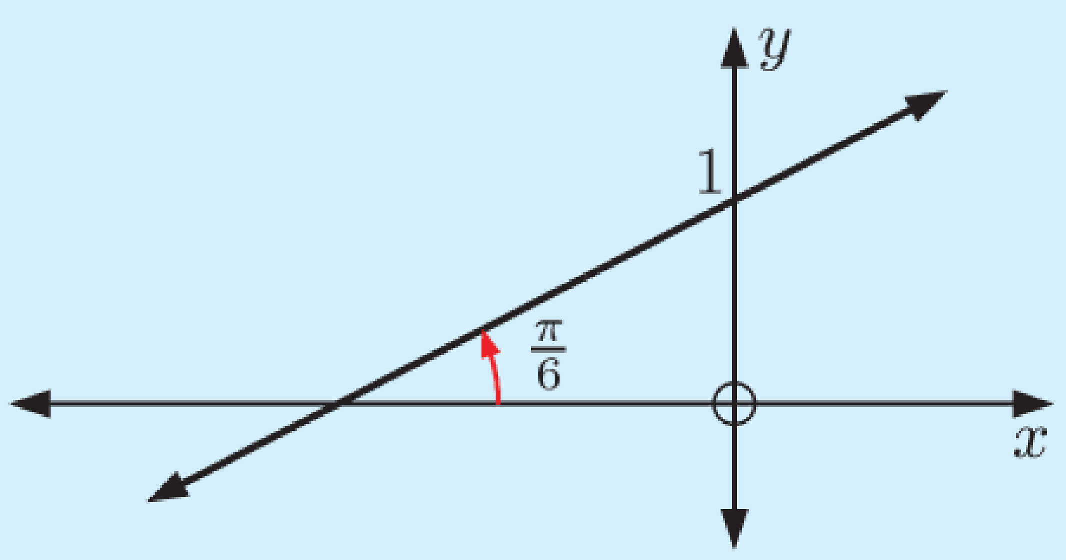
- For $m < 0$:



$$\text{Gradient } m = \frac{0 - b}{a - 0} = -\frac{b}{a} = -\tan(\pi - \theta) = \tan\theta$$

Example 13 — Self Tutor

Find the equation of the given line:



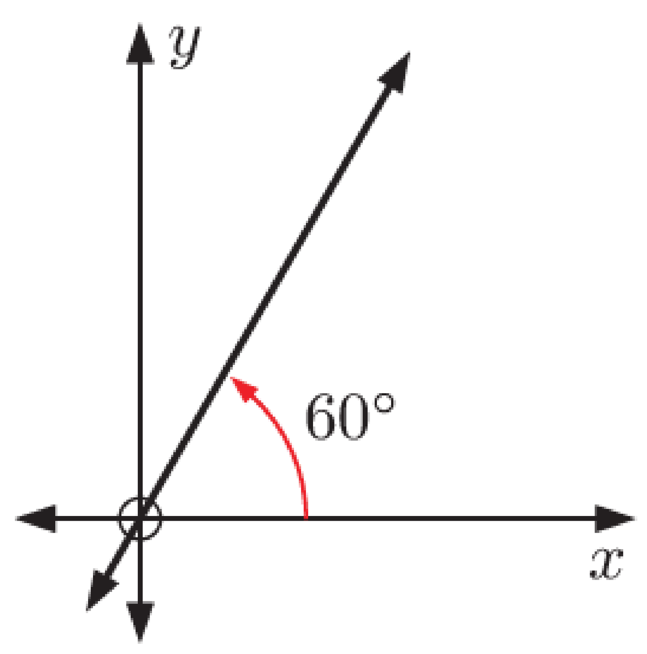
The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and y-intercept 1.

$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + 1$.

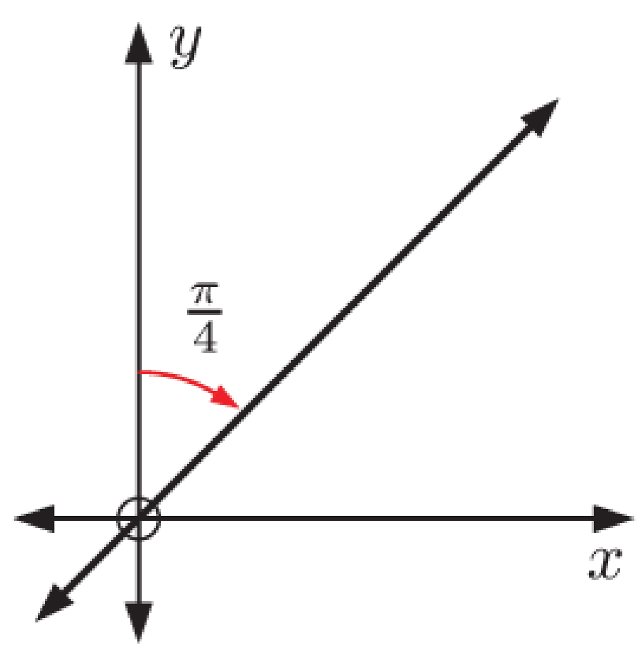
EXERCISE 8G

1 Find the equation of each line:

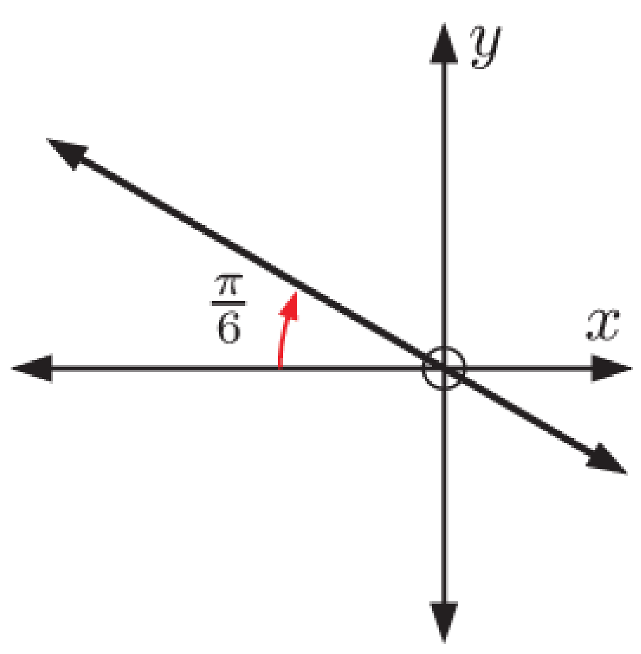
a



b

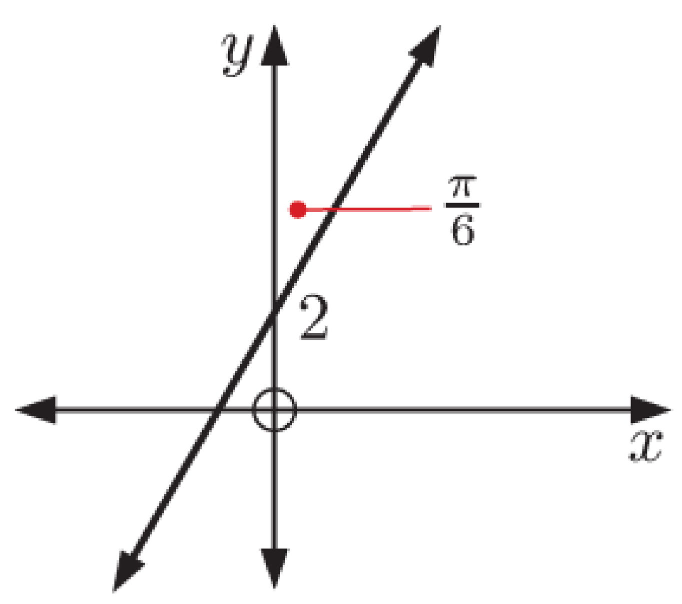


c

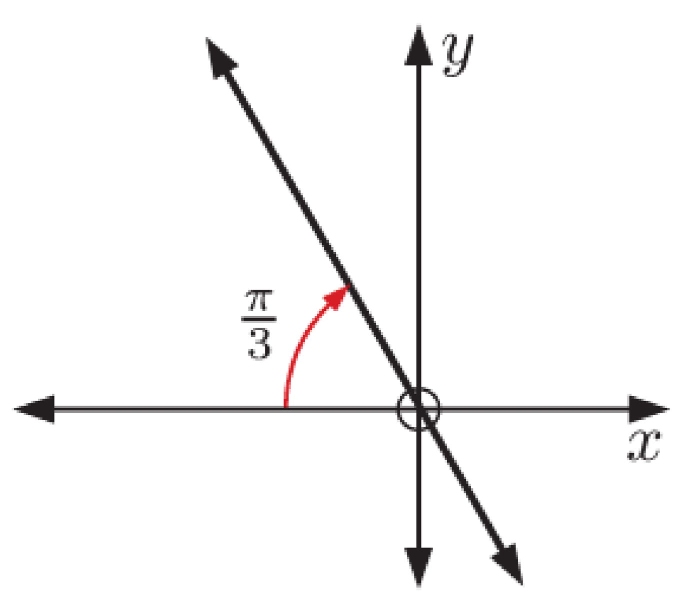


2 Find the equation of each line:

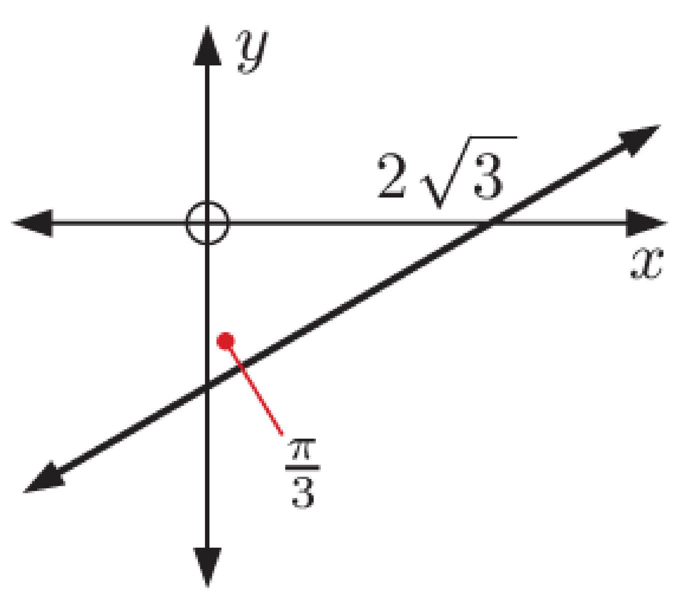
a



b

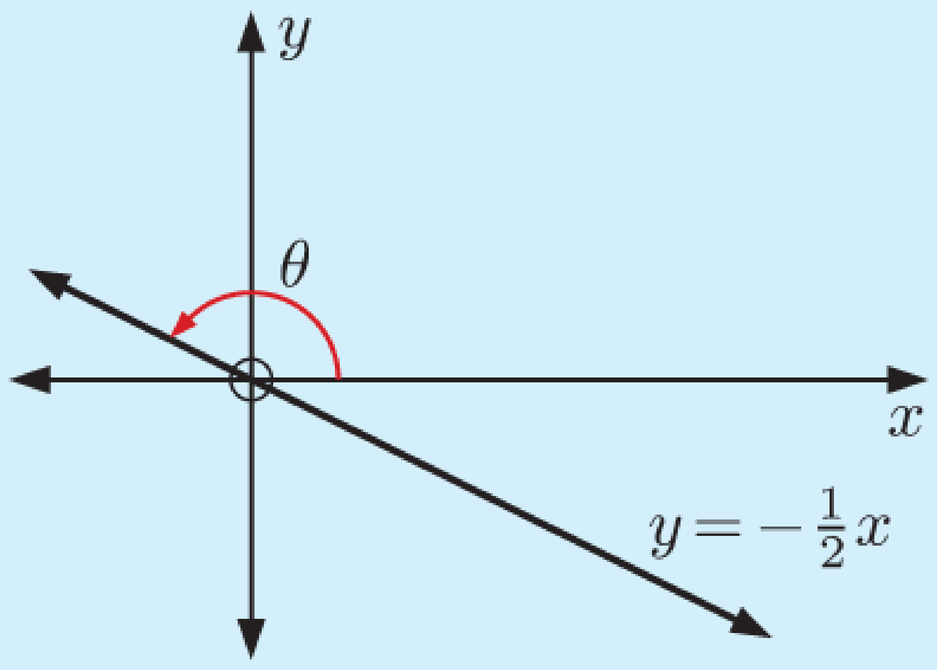


c



Example 14 — Self Tutor

Find, in radians, the measure of θ:



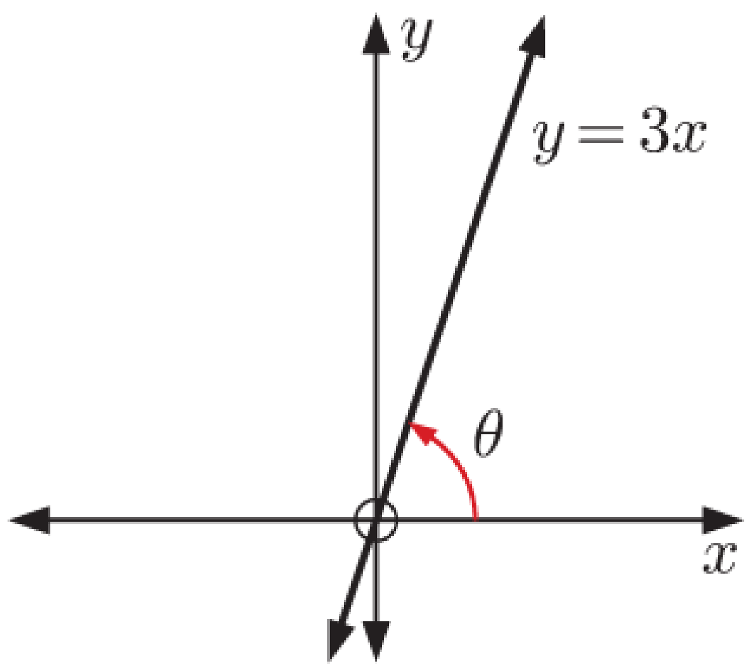
The line has gradient $-\frac{1}{2}$, so $\tan\theta = -\frac{1}{2}$.

Using technology, $\tan^{-1}(-\frac{1}{2}) \approx -0.464$

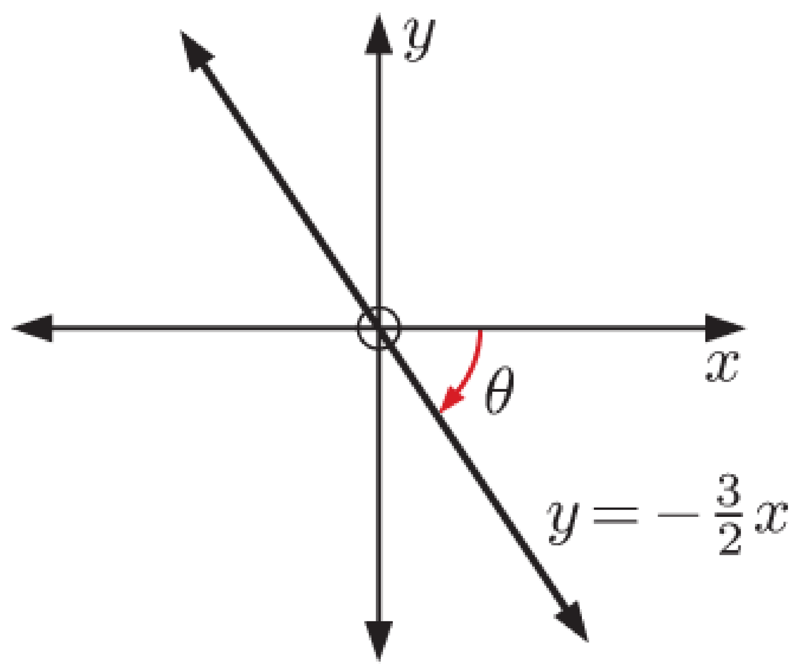
But $0 < \theta < \pi$, so $\theta \approx \pi - 0.464 \approx 2.68$

3 Find, in radians, the measure of θ:

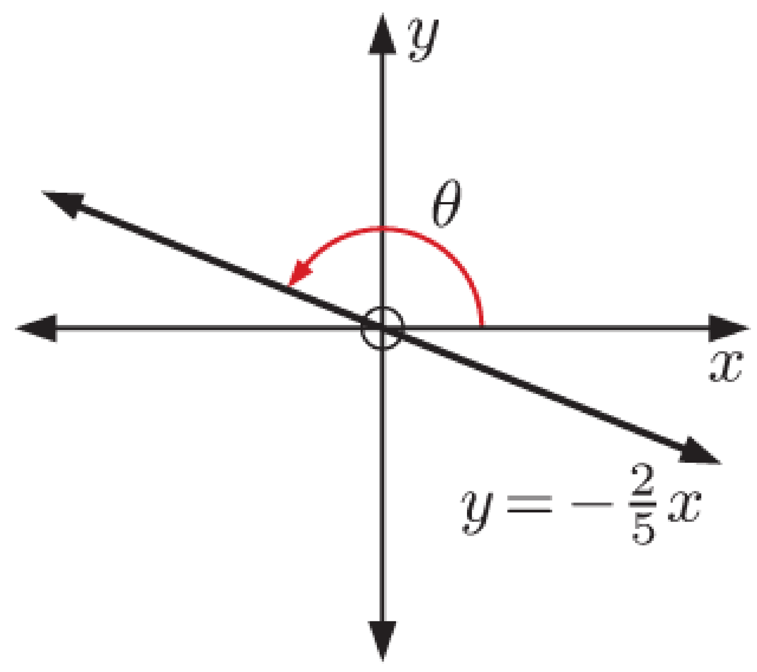
a



b

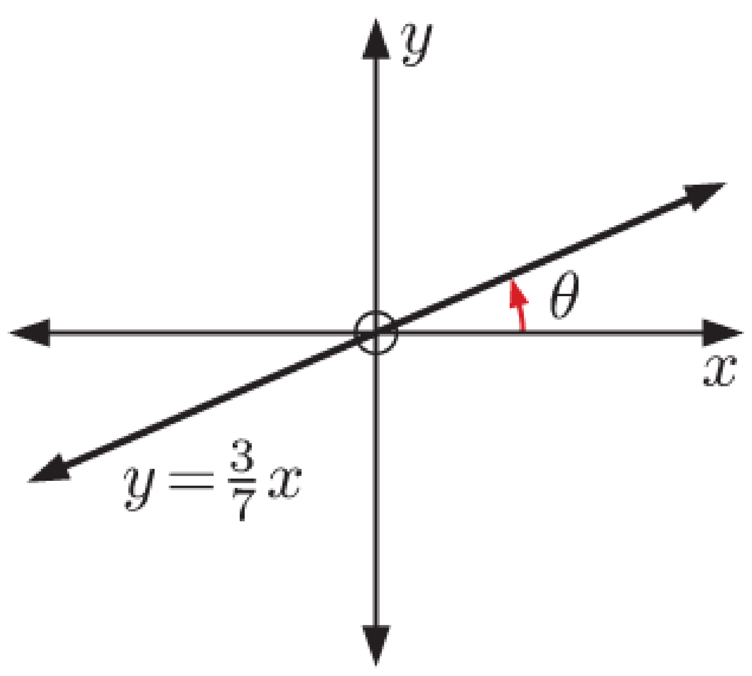


c

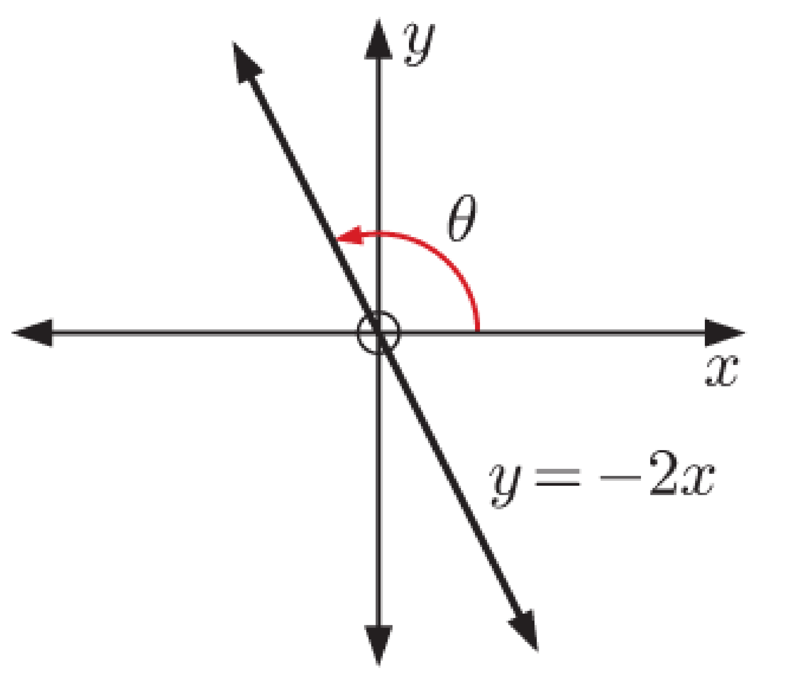


4 Find, in degrees, the measure of θ:

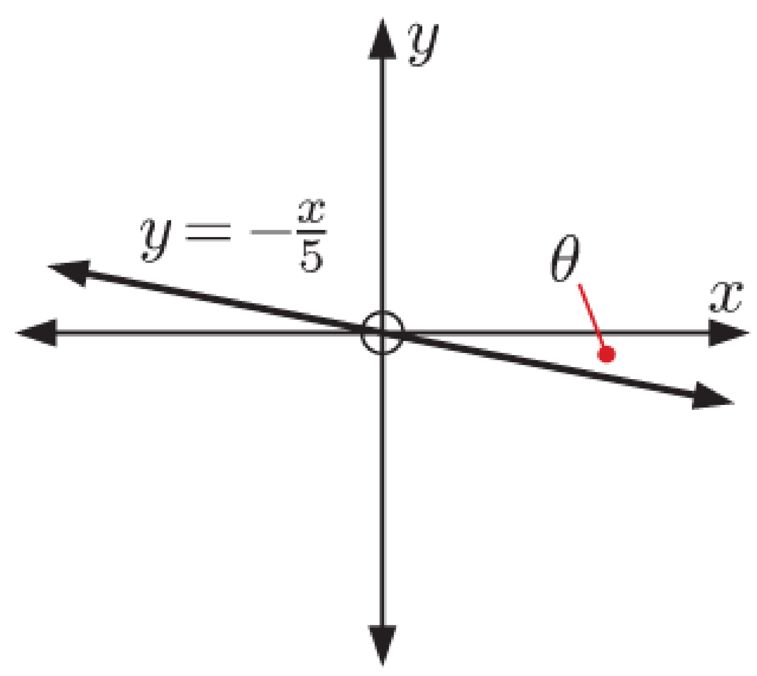
a



b



c



REVIEW SET 8A

1 Convert to radians in terms of π:

a $120°$ **b** $225°$ **c** $150°$ **d** $540°$

2 Illustrate the quadrants where $\sin\theta$ and $\cos\theta$ have the same sign.

3 Determine the coordinates of the point on the unit circle corresponding to an angle of:

a $320°$ **b** $163°$ **c** 0.68^c **d** $\frac{11\pi}{6}$

4 Find the arc length of a sector with angle 1.5 radians and radius 8 cm.

5 Find the acute angles that have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as $165°$ **c** cosine as $276°$.

6 Find:

a $\sin 159°$ if $\sin 21° \approx 0.358$
b $\cos 92°$ if $\cos 88° \approx 0.035$
c $\cos 75°$ if $\cos 105° \approx -0.259$
d $\tan(-133°)$ if $\tan 47° \approx 1.072$.

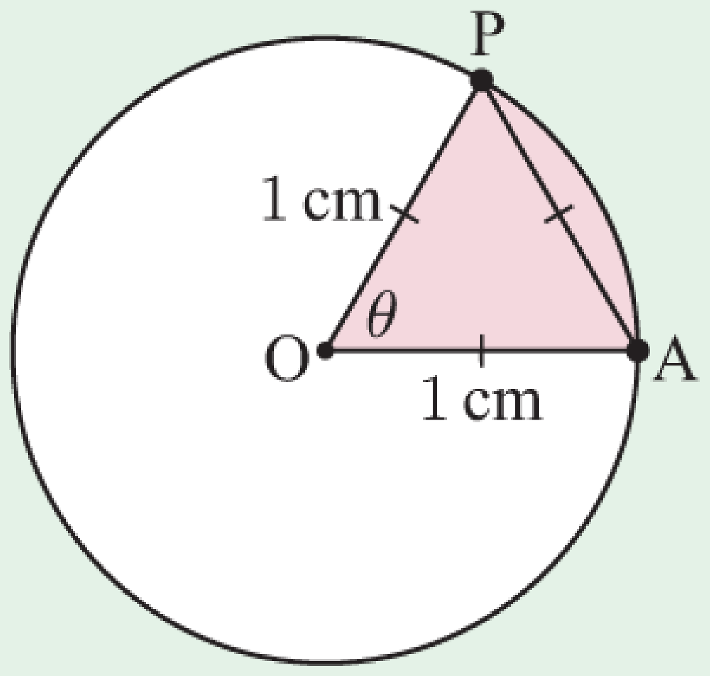
7 Use a unit circle diagram to find:

a $\cos 360°$ and $\sin 360°$
b $\cos(-\pi)$ and $\sin(-\pi)$.

8 Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{2\pi}{3}$
b $\frac{8\pi}{3}$

9



a State the value of θ in:
  i degrees
  ii radians.

b State the arc length AP.

c State the area of the minor *sector* OAP.

10 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ exactly.

11 If $\cos\theta = \frac{3}{4}$, find the possible values of $\sin\theta$.

12 Evaluate:

a $2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$
b $\tan^2\left(\frac{\pi}{4}\right) - 1$
c $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

13 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find:

a $\cos x$
b $\sin x$.

14 Suppose $\cos\theta = \frac{\sqrt{11}}{\sqrt{17}}$ and θ is acute. Find the exact value of $\tan\theta$.

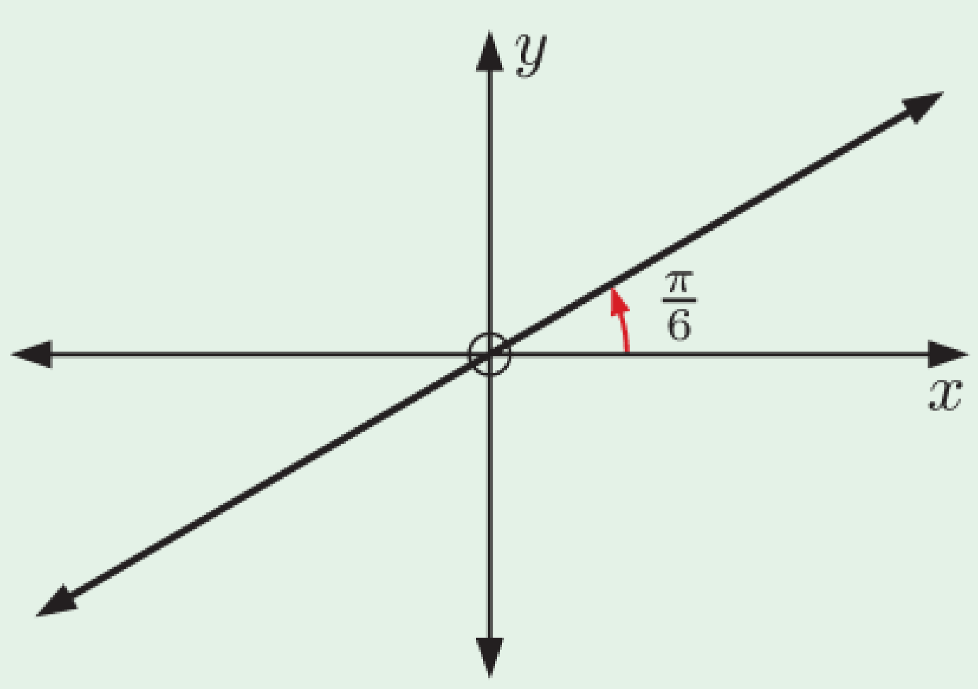
15 Explain how to use the unit circle to find θ when $\cos\theta = -\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

16 Find two angles on the unit circle with $0 \leqslant \theta \leqslant 2\pi$, such that:

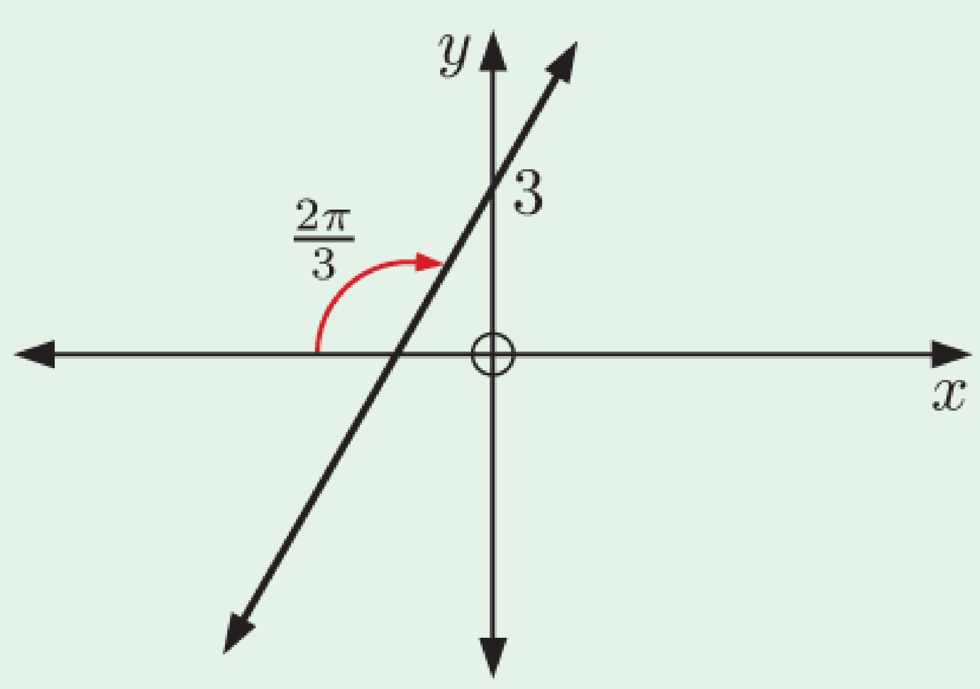
a $\cos\theta = \frac{2}{3}$
b $\sin\theta = -\frac{1}{4}$
c $\tan\theta = 3$

17 Find the equation of each line:

a



b



REVIEW SET 8B

1 Convert to degrees, to 2 decimal places:

a $\frac{2\pi}{5}$
b 1.46
c 0.435^c
d -5.271

2 Determine the area of a sector with angle $\frac{5\pi}{12}$ and radius 13 cm.

3 Find the angle [OA] makes with the positive x-axis if the x-coordinate of the point A on the unit circle is -0.222.

4 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

5 A sector has perimeter 21 cm and area 27 cm^2. Find the radius of the sector.

6 Use a unit circle diagram to find:

a $\cos\frac{3\pi}{2}$ and $\sin\frac{3\pi}{2}$ **b** $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$

7 Suppose $m = \sin p$, where p is acute. Write an expression in terms of m for:

a $\sin(\pi - p)$ **b** $\sin(p + 2\pi)$ **c** $\cos p$ **d** $\tan p$

8 Find all angles between $0°$ and $360°$ which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a sine of $\frac{1}{\sqrt{2}}$ **c** a tangent of $-\sqrt{3}$

9 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

10 Find the obtuse angles which have the same:

a sine as $47°$ **b** sine as $\frac{\pi}{15}$ **c** cosine as $186°$

11 Find the perimeter and area of a sector with radius 11 cm and angle $63°$.

12 Show that $\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} = -\sqrt{2}$.

13 If $\cos\theta = -\frac{3}{4}$, $\frac{\pi}{2} < \theta < \pi$ find the exact value of:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi - \theta)$

14 Without using a calculator, evaluate:

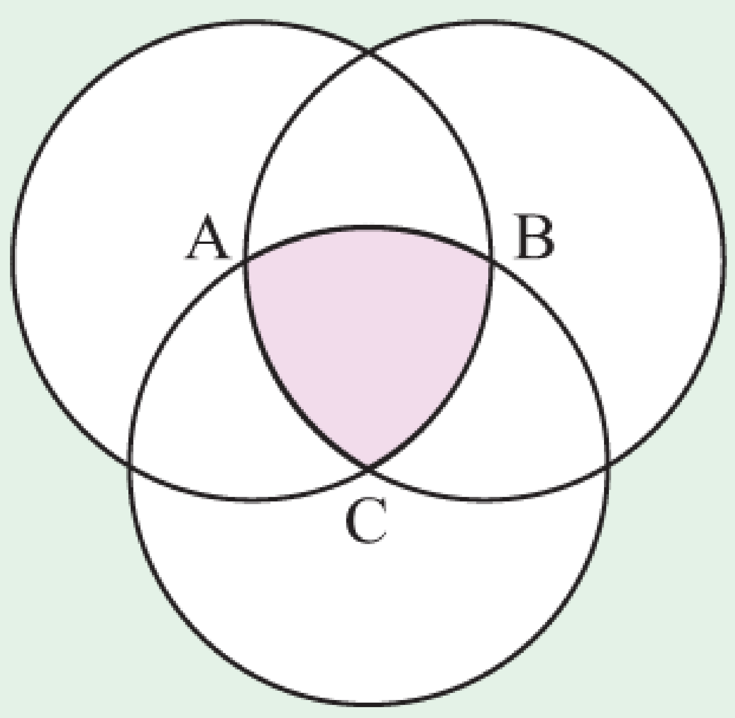
a $\tan^2 60° - \sin^2 45°$ **b** $\cos^2\left(\frac{\pi}{4}\right) + \sin\frac{\pi}{2}$

c $\cos\frac{5\pi}{3} - \tan\frac{5\pi}{4}$ **d** $\tan^2\left(\frac{2\pi}{3}\right)$

15 Use a unit circle diagram to show that $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ for $\frac{\pi}{2} < \theta < \pi$.

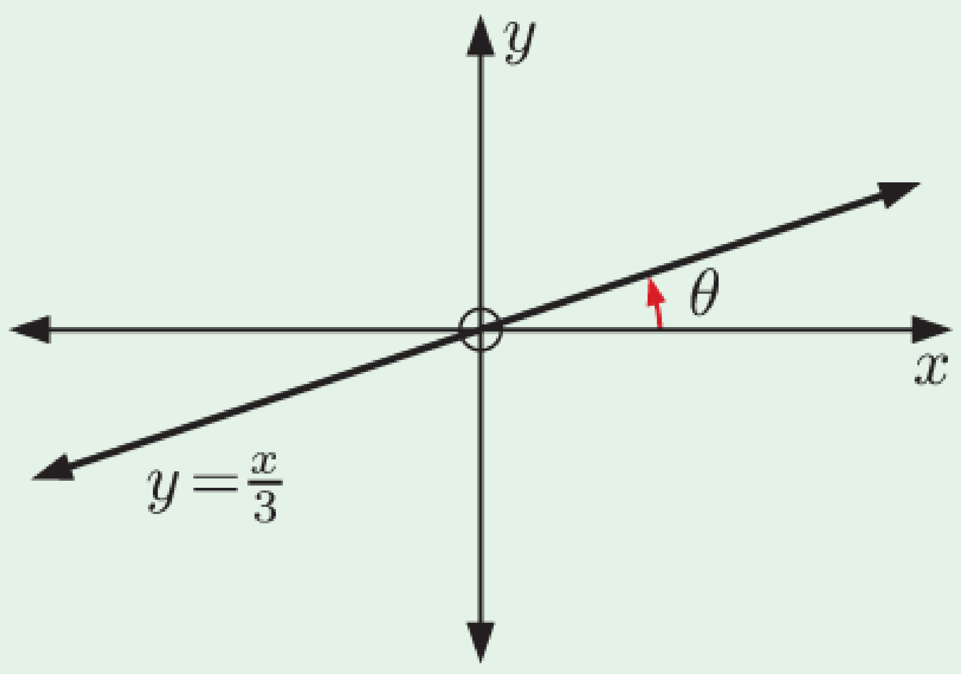
16 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is $A = \frac{r^2}{2}(\pi - \sqrt{3})$.

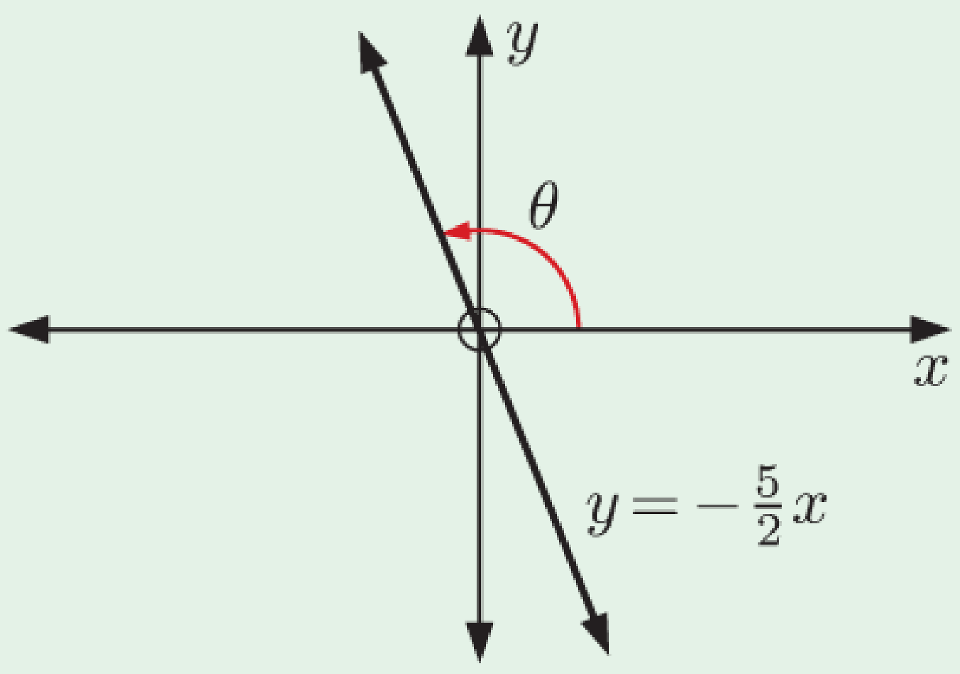


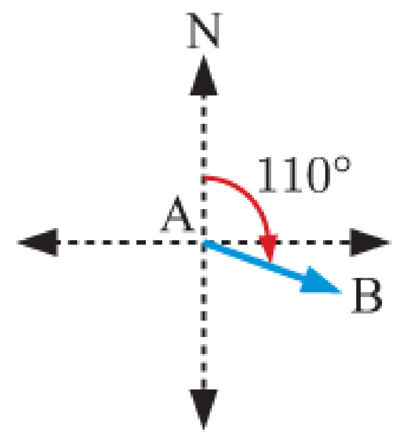
17 Find, in radians, the measure of θ:

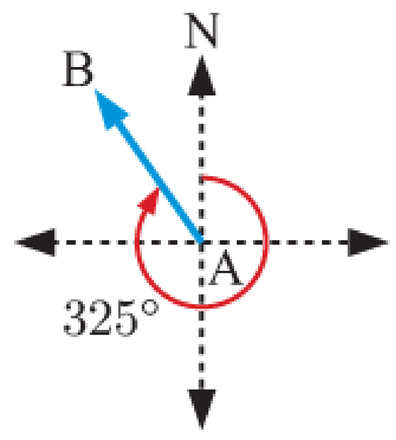
a



b



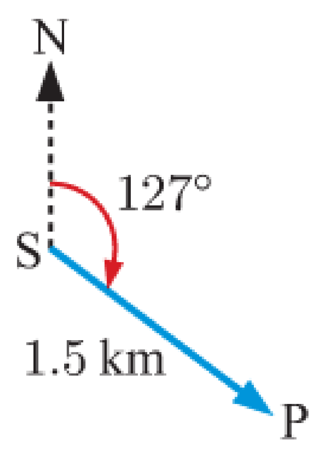
c 

d 

2 **a** 126° **b** 245° **c** 152° **d** 308°

3 **a** 072° **b** 252° **c** 162° **d** 342°
e 113° **f** 293°

4 ≈ 125° **5** **a** ≈ 224 m **b** ≈ 333° **c** ≈ 153°

6 **a** 

b ≈ 1.20 km **c** ≈ 0.903 km

7 ≈ 2.41 km **8** ≈ 12.6 km

9 **a** ≈ 854 m **b** ≈ 203°

10 ≈ 73.3 km on the bearing ≈ 191°

11 ≈ 17.8 km on the bearing ≈ 162°

12 **a** ≈ 046.6° **b** ≈ 4.22 km

EXERCISE 7F

1 **a** **i** [EH] **ii** [EF] **iii** [EG] **iv** [FH]
b **i** [MR] **ii** [MN]

2 **a** **i** $\widehat{\text{AFE}}$ **ii** $\widehat{\text{BMF}}$ **iii** $\widehat{\text{ADE}}$ **iv** $\widehat{\text{BNF}}$
b **i** $\widehat{\text{BAM}}$ **ii** $\widehat{\text{BNM}}$ **iii** $\widehat{\text{EAN}}$

3 **a** **i** ≈ 36.9° **ii** ≈ 25.1° **iii** ≈ 56.3° **iv** ≈ 29.1°
b **i** ≈ 33.7° **ii** ≈ 33.7° **iii** ≈ 25.2° **iv** ≈ 30.8°
c **i** ≈ 59.0° **ii** ≈ 22.0° **iii** ≈ 22.6°
d **i** ≈ 64.9° **ii** ≈ 71.7°

4 ≈ 31.7°

REVIEW SET 7A

1 **a** 10 cm **b** $\frac{6}{10} = \frac{3}{5}$ **c** $\frac{8}{10} = \frac{4}{5}$ **d** $\frac{6}{8} = \frac{3}{4}$

2 **a** $x \approx 3.51$ **b** $x \approx 51.1$ **c** $x \approx 5.62$

3 ≈ 43.4 cm^2 **4** $\theta = 33°$, $x \approx 3.90$, $y \approx 7.15$

5 $\theta \approx 8.19°$ **6** ≈ 124°

7 **a** $x \approx 2.8$ **b** $x \approx 4.2$ **c** $x \approx 5.2$

8 ≈ 13.5 m **9** **a** 118° **b** 231° **c** 329°

10 13 km on the bearing ≈ 203° from the helipad.

11 ≈ 8.74° **12** ≈ 0.607 L **13** **a** ≈ 53.1° **b** ≈ 62.1°

REVIEW SET 7B

1 **a** AB ≈ 4.5 cm, AC ≈ 2.2 cm, BC ≈ 5.0 cm
b **i** ≈ 0.44 **ii** ≈ 0.90 **iii** ≈ 0.49

2 **a** $\theta \approx 34.8°$ **b** $\theta \approx 39.7°$ **c** $\theta \approx 36.0°$

3 AB ≈ 120 mm, AC ≈ 111 mm

4 $x \approx 25.7$, $\theta \approx 53.6°$, $\alpha \approx 36.4°$

5 **a** ≈ 200 cm **b** ≈ 1500 cm^2 **6** ≈ 2.54 cm

7 ≈ 204 m **8** **a** 90° **b** ≈ 33.9°

9 ≈ 3.91 km on the bearing ≈ 253° from his starting point.

10 ≈ 5.46 km **11** ≈ 485 m^3

12 **a** ≈ 14.4° **b** ≈ 18.9° **c** ≈ 21.8°

13 **a** **i** ≈ 27.6 cm **ii** ≈ 23.3 cm **b** ≈ 6010 cm^3

EXERCISE 8A

1 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{3}$ **c** $\frac{\pi}{6}$ **d** $\frac{\pi}{10}$ **e** $\frac{\pi}{20}$
f $\frac{3\pi}{4}$ **g** $\frac{5\pi}{4}$ **h** $\frac{3\pi}{2}$ **i** 2π **j** 4π
k $\frac{7\pi}{4}$ **l** 3π **m** $\frac{\pi}{5}$ **n** $\frac{4\pi}{9}$ **o** $\frac{23\pi}{18}$

2 **a** $\approx 0.641^c$ **b** $\approx 2.39^c$ **c** $\approx 5.55^c$ **d** $\approx 3.83^c$
e $\approx 6.92^c$

3 **a** 36° **b** 108° **c** 135° **d** 10° **e** 20°
f 140° **g** 18° **h** 27° **i** 210° **j** 22.5°

4 **a** ≈ 114.59° **b** ≈ 87.66° **c** ≈ 49.68°
d ≈ 182.14° **e** ≈ 301.78°

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Deg.	0	30	60	90	120	150	180	210	240	270	300	330	360
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

1 **a** 7 cm **b** 12 cm **c** ≈ 13.0 m

2 **a** 6 cm^2 **b** 48 cm^2 **c** ≈ 8.21 cm^2

3 **a** arc length ≈ 49.5 cm, area ≈ 223 cm^2
b arc length ≈ 23.0 cm, area ≈ 56.8 cm^2

4 **a** $\approx 0.686^c$ **b** 0.6^c

5 **a** $\theta = 0.75^c$, area = 24 cm^2
b $\theta = 1.68^c$, area = 21 cm^2
c $\theta \approx 2.32^c$, area = 126.8 cm^2

6 **a** ≈ 3.15 m **b** ≈ 9.32 m^2

7 **a** ≈ 5.91 cm **b** ≈ 18.9 cm

8 **a** $\alpha \approx 0.3218^c$ **b** $\theta \approx 2.498^c$ **c** ≈ 387 m^2

9 **a** ≈ 11.7 cm **b** $r \approx 11.7$ **c** ≈ 37.7 cm **d** $\theta \approx 3.23^c$

10 ≈ 25.9 cm **11** **b** ≈ 2 h 24 min **12** ≈ 227 m^2

13 **a** $\alpha \approx 5.739$ **b** $\theta \approx 168.5$ **c** $\phi \approx 191.5$
d ≈ 71.62 cm

14 **a** 4 cm **b** **i** ≈ 2.16 cm^2 **ii** ≈ 29.3 cm^2

15 **a** **Hint:** Let the largest circle have radius r_1, and use a right angled triangle to show that $\sin\frac{\pi}{6} = \frac{r_1}{10 - r_1}$.
b $\frac{25\pi}{2}$ units^2 **c** $\frac{3}{4}$

EXERCISE 8C

1

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	−1	0	1
cosine	1	0	−1	0	1	0
tangent	0	undef.	0	undef.	0	undef.

2 **a** **i** A(cos 26°, sin 26°), B(cos 146°, sin 146°), C(cos 199°, sin 199°)
ii A(0.899, 0.438), B(−0.829, 0.559), C(−0.946, −0.326)
b **i** A(cos 123°, sin 123°), B(cos 251°, sin 251°), C(cos(−35°), sin(−35°))
ii A(−0.545, 0.839), B(−0.326, −0.946), C(0.819, −0.574)

3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

5 **a** 1 and 4 **b** 2 and 3 **c** 3 **d** 2

6 **a** $\cos 400° = \cos(360 + 40)° = \cos 40°$

b $\sin\frac{5\pi}{7} = \sin\left(\frac{5\pi}{7} + 2\pi\right) = \sin\frac{19\pi}{7}$

c $\tan\frac{13\pi}{8} = \tan\left(\frac{13\pi}{8} - 3\pi\right) = \tan\left(-\frac{11\pi}{8}\right)$

7 **B** and **D** **8** **B** and **E**

9 **a** **i** ≈ 0.985 **ii** ≈ 0.985 **iii** ≈ 0.866 **iv** ≈ 0.866

v 0.5 **vi** 0.5 **vii** ≈ 0.707 **viii** ≈ 0.707

b $\sin(180° - \theta) = \sin\theta$ **c** $\sin(\pi - \theta) = \sin\theta$

d The points have the same y-coordinate.

e **i** $135°$ **ii** $129°$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{5\pi}{6}$

10 **a** **i** ≈ 0.342 **ii** ≈ -0.342 **iii** 0.5

iv -0.5 **v** ≈ 0.906 **vi** ≈ -0.906

vii ≈ 0.174 **viii** ≈ -0.174

b $\cos(180° - \theta) = -\cos\theta$ **c** $\cos(\pi - \theta) = -\cos\theta$

d The x-coordinates of the points have the same magnitude but are opposite in sign.

e **i** $140°$ **ii** $161°$ **iii** $\frac{4\pi}{5}$ **iv** $\frac{3\pi}{5}$

11 $\tan(\pi - \theta) = -\tan\theta$

12 **a** ≈ 0.6820 **b** ≈ 0.8572 **c** ≈ -0.7986

d ≈ 0.9135 **e** ≈ 0.9063 **f** ≈ -0.6691

13 **a**

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	≈ 0.682	≈ -0.682	≈ 0.732	≈ 0.732
1.772	≈ 0.980	≈ -0.980	≈ -0.200	≈ -0.200
3.414	≈ -0.269	≈ 0.269	≈ -0.963	≈ -0.963
6.25	≈ -0.0332	≈ 0.0332	≈ 0.999	≈ 0.999
−1.17	≈ -0.921	≈ 0.921	≈ 0.390	≈ 0.390

b $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

c Q has coordinates $(\cos(-\theta), \sin(-\theta))$ or $(\cos\theta, -\sin\theta)$ (since it is the reflection of P in the x-axis)

$\therefore$ $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

d $\cos(2\pi - \theta) = \cos(-\theta) = \cos\theta$

$\sin(2\pi - \theta) = \sin(-\theta) = -\sin\theta$

e $\tan(2\pi - \theta) = -\tan\theta$

14 **a** The angle between [OP] and the positive x-axis is $\left(\frac{\pi}{2} - \theta\right)$.

$\therefore$ P is $\left(\cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right)\right)$

b **i** In $\triangle$OXP, $\sin\theta = \frac{XP}{OP} = \frac{XP}{1}$

$\therefore$ $XP = \sin\theta$

ii In $\triangle$OXP, $\cos\theta = \frac{OX}{OP} = \frac{OX}{1}$

$\therefore$ $OX = \cos\theta$

c **i** $\cos\left(\frac{\pi}{2} - \theta\right) = XP = \sin\theta$

ii $\sin\left(\frac{\pi}{2} - \theta\right) = OX = \cos\theta$

d **i** $\cos\frac{\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \sin\frac{3\pi}{10} \approx 0.809$

ii $\sin\frac{\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{3\pi}{8} \approx 0.383$

e $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$

EXERCISE 8D

1

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	-1	-1	0	1

2

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

b $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $\tan\left(-\frac{\pi}{4}\right) = -1$

4 **a** $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$ **b** $\tan\frac{\pi}{2}$ is undefined

5 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{4}$ **d** $-\frac{1}{4}$ **e** 1 **f** $\sqrt{2}$

g $\frac{1}{2}$ **h** $\frac{1}{2}$ **i** 2 **j** -1 **k** $-\sqrt{3}$ **l** $-\sqrt{3}$

6 **a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$ **d** $\frac{2\pi}{3}, \frac{4\pi}{3}$

e $\frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\frac{4\pi}{3}, \frac{5\pi}{3}$

7 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$ **d** $0, \pi, 2\pi$

e $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

8 **a** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **c** $\frac{3\pi}{2}, \frac{7\pi}{2}$

9 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$

d $\theta = \frac{\pi}{2}$ **e** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

g $\theta = 0, \pi, 2\pi$ **h** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

i $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ **j** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $\theta = k\pi$, $k \in \mathbb{Z}$ **b** $\theta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

EXERCISE 8E

1 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$

d $\cos\theta = 0$

2 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$

d $\sin\theta = \pm 1$

3 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$

d $\sin\theta = -\frac{12}{13}$

4 **a** $\tan\theta = -\frac{1}{2\sqrt{2}}$ **b** $\tan\theta = -2\sqrt{6}$ **c** $\tan\theta = \frac{1}{\sqrt{2}}$
d $\tan\theta = -\frac{\sqrt{7}}{3}$

5 **a** $\sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$ **b** $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$
c $\sin\theta = -\sqrt{\frac{5}{14}}$, $\cos\theta = -\frac{3}{\sqrt{14}}$
d $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$

6 $\sin\theta = \dfrac{-k}{\sqrt{k^2+1}}$, $\cos\theta = \dfrac{-1}{\sqrt{k^2+1}}$

EXERCISE 8F

1 **a** $\theta \approx 76.0°$ or $256°$ **b** $\theta \approx 33.9°$ or $326.1°$
c $\theta \approx 36.9°$ or $143.1°$ **d** $\theta = 90°$ or $270°$
e $\theta \approx 81.5°$ or $261.5°$ **f** $\theta \approx 83.2°$ or $276.8°$

2 **a** $\theta \approx 0.322$ or 3.46 **b** $\theta \approx 1.13$ or 5.16
c $\theta \approx 0.656$ or 2.49 **d** $\theta \approx 1.32$ or 4.97
e $\theta \approx 0.114$ or 3.26 **f** $\theta \approx 0.167$ or 2.97

3 **a** $\theta \approx 1.82$ or 4.46 **b** $\theta = 0, \pi$, or 2π
c $\theta \approx 1.88$ or 5.02 **d** $\theta \approx 3.58$ or 5.85
e $\theta \approx 0.876$ or 4.02 **f** $\theta \approx 0.674$ or 5.61
g $\theta \approx 0.0910$ or 3.05 **h** $\theta \approx 2.19$ or 4.10

4 **a** $\theta \approx -95.7°$ or $95.7°$ **b** $\theta \approx 53.1°$ or $126.9°$
c $\theta \approx -56.3°$ or $123.7°$ **d** $\theta \approx -36.9°$ or $36.9°$
e $\theta \approx -39.8°$ or $140.2°$ **f** $\theta \approx -140.5°$ or $-39.5°$

5 **a** $\theta \approx 1.27$ or 5.02
b For $\theta \approx 1.27$: $\sin\theta = \frac{\sqrt{91}}{10}$, $\tan\theta = \frac{\sqrt{91}}{3}$
For $\theta \approx 5.02$: $\sin\theta = -\frac{\sqrt{91}}{10}$, $\tan\theta = -\frac{\sqrt{91}}{3}$

EXERCISE 8G

1 **a** $y = \sqrt{3}x$ **b** $y = x$ **c** $y = -\frac{1}{\sqrt{3}}x$

2 **a** $y = \sqrt{3}x + 2$ **b** $y = -\sqrt{3}x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

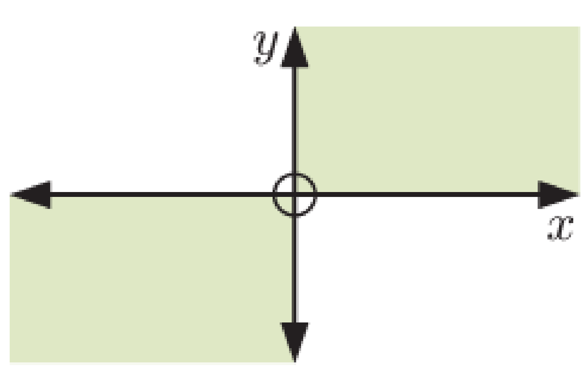
3 **a** $\theta \approx 1.25$ **b** $\theta \approx -0.983$ **c** $\theta \approx -0.381$

4 **a** $\theta \approx 23.2°$ **b** $\theta \approx 117°$ **c** $\theta \approx -11.3°$

REVIEW SET 8A

1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π

2



3 **a** $(0.766, -0.643)$ **b** $(-0.956, 0.292)$
c $(0.778, 0.629)$ **d** $(0.866, -0.5)$

4 12 cm **5** **a** $\frac{\pi}{3}$ **b** $15°$ **c** $84°$

6 **a** ≈ 0.358 **b** ≈ -0.035 **c** ≈ 0.259 **d** ≈ 1.072

7 **a** $\cos 360° = 1$, $\sin 360° = 0$
b $\cos(-\pi) = -1$, $\sin(-\pi) = 0$

8 **a** $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$
b $\sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{8\pi}{3} = -\frac{1}{2}$, $\tan\frac{8\pi}{3} = -\sqrt{3}$

9 **a** **i** $60°$ **ii** $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ cm **c** $\frac{\pi}{6}$ cm^2

10 $\tan x = \frac{1}{\sqrt{15}}$ **11** $\sin\theta = \pm\frac{\sqrt{7}}{4}$

12 **a** $\frac{\sqrt{3}}{2}$ **b** 0 **c** $\frac{1}{2}$ **13** **a** $\frac{2}{\sqrt{13}}$ **b** $-\frac{3}{\sqrt{13}}$

14 $\tan\theta = \frac{\sqrt{6}}{\sqrt{11}}$

16 **a** $\theta \approx 0.841$ or 5.44 **b** $\theta \approx 3.39$ or 6.03
c $\theta \approx 1.25$ or 4.39

17 **a** $y = \frac{1}{\sqrt{3}}x$ **b** $y = \sqrt{3}x + 3$

REVIEW SET 8B

1 **a** $72°$ **b** $\approx 83.65°$ **c** $\approx 24.92°$ **d** $\approx -302.01°$

2 ≈ 111 cm^2 **3** $\approx 103°$

4 radius ≈ 8.79 cm, area ≈ 81.0 cm^2 **5** 4.5 cm or 6 cm

6 **a** $\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$
b $\cos\left(-\frac{\pi}{2}\right) = 0$, $\sin\left(-\frac{\pi}{2}\right) = -1$

7 **a** $\sin(\pi - p) = m$ **b** $\sin(p + 2\pi) = m$
c $\cos p = \sqrt{1-m^2}$ **d** $\tan p = \dfrac{m}{\sqrt{1-m^2}}$

8 **a** $150°, 210°$ **b** $45°, 135°$ **c** $120°, 300°$

9 **a** $\theta = \pi$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $133°$ **b** $\frac{14\pi}{15}$ **c** $174°$

11 perimeter ≈ 34.1 cm, area ≈ 66.5 cm^2

13 **a** $\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$ **c** $\frac{3}{4}$

14 **a** $2\frac{1}{2}$ **b** $1\frac{1}{2}$ **c** $-\frac{1}{2}$ **d** 3

17 **a** $\theta \approx 0.322$ **b** $\theta \approx 1.95$

EXERCISE 9A

1 **a** ≈ 28.9 cm^2 **b** ≈ 384 km^2 **c** 20 m^2

2 **a** ≈ 18.7 cm^2 **b** ≈ 28.3 cm^2 **c** ≈ 52.0 m^2

3 $x \approx 19.0$ **4** **a** ≈ 166 cm^2 **b** ≈ 1410 cm^2

5 ≈ 18.9 cm^2 **6** ≈ 137 cm^2

7 **a** ≈ 71.616 m^2 **b** ≈ 8.43 m

8 ≈ 374 cm^2 **9** ≈ 7.49 cm **10** ≈ 11.9 m

11 **a** $\approx 48.6°$ or $\approx 131.4°$ **b** $\approx 42.1°$ or $\approx 137.9°$

12 $\frac{1}{4}$ is not covered

13 **a** ≈ 36.2 cm^2 **b** ≈ 62.8 cm^2 **c** ≈ 40.4 mm^2
d ≈ 19.3 cm^2

14 ≈ 4.69 cm^2

EXERCISE 9B

1 **a** ≈ 28.8 cm **b** ≈ 3.38 km **c** ≈ 14.2 m

2 **a** $\theta \approx 82.8°$ **b** $\theta \approx 54.8°$ **c** $\theta \approx 98.2°$

3 $B\widehat{A}C \approx 52.0°$, $A\widehat{B}C \approx 59.3°$, $A\widehat{C}B \approx 68.7°$

4 **a** $\approx 112°$ **b** ≈ 16.2 cm^2

5 **a** $\approx 40.3°$ **b** $\approx 107°$

6 **a** $\cos\theta = 0.65$ **b** $x \approx 3.81$

7 **a** $\theta \approx 75.2°$ **b** ≈ 6.30 m

8 **a** DB ≈ 4.09 m, BC ≈ 9.86 m
b $A\widehat{B}E \approx 68.2°$, $D\widehat{B}C \approx 57.5°$ **c** ≈ 17.0 m^2

9 **b** $x = 3 + \sqrt{22}$

10 **a** $x \approx 10.8$ **b** $x \approx 2.77$ **c** $x \approx 2.89$

11 $x \approx 1.41$ or 7.78 **12** BD ≈ 12.4 cm

13 $\theta \approx 71.6°$ **14** ≈ 6.40 cm

15 **a** $x = 2$ **b** $4\sqrt{6}$ cm^2 **16** $\approx 63°, 117°, 36°, 144°$

EXERCISE 9C.1

1 **a** $x \approx 28.4$ **b** $x \approx 13.4$ **c** $x \approx 3.79$
d $x \approx 10.3$ **e** $x \approx 4.49$ **f** $x \approx 7.07$