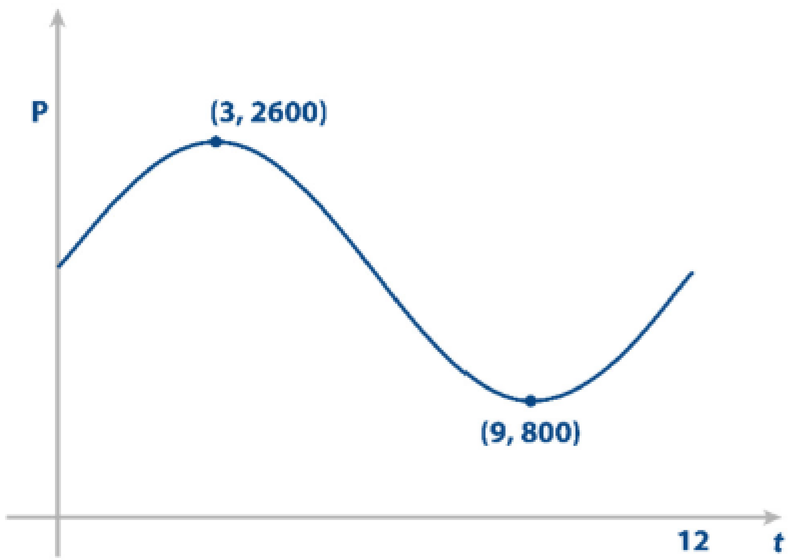
Trig modelling - exam questions [60 marks]

1. [Maximum mark: 7] EXN.1.SL.TZ0.6

The size of the population (P) of migrating birds in a particular town can be approximately modelled by the equation $P = a\sin(bt) + c,\ \ a, b, c \in \mathbb{R}^+$, where t is measured in months from the time of the initial measurements.

In a 12 month period the maximum population is 2600 and occurs when $t = 3$ and the minimum population is 800 and occurs when $t = 9$.

This information is shown on the graph below.



(a.i) Find the value of a. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\frac{2600-800}{2} = 900$ **(M1)A1**

[2 marks]

(a.ii) Find the value of b. [2]

Markscheme

$\frac{360}{12} = 30$ **(M1)A1**

Note: Accept $\frac{2\pi}{12} = 0.524\ (0.523598\ldots)$.

[2 marks]

(a.iii) Find the value of c. [1]

Markscheme

$\frac{2600+800}{2} = 1700$ **A1**

[1 mark]

(b) Find the value of t at which the population first reaches 2200. [2]

Markscheme

Solve $900\sin(30t) + 1700 = 2200$ **(M1)**

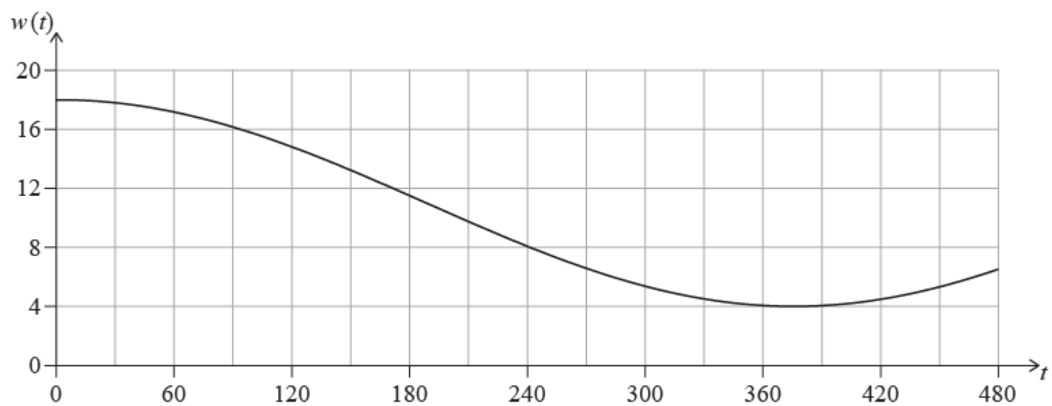
$t = 1.12\ (1.12496\ldots)$ **A1**

[2 marks]

2. [Maximum mark: 15] 23M.2.SL.TZ1.3

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^{\circ}) + d$ where t is the length of time, in minutes, after $06:00$.

On 20 January, the first high tide occurs at $06:00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12:15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

Markscheme

$\frac{18-4}{2}$ *(M1)*

$(a =)\ 7$ ***A1***

[2 marks]

(b) Find the value of d. [2]

Markscheme

$\frac{18+4}{2}$ **OR** $18-7$ **OR** $4+7$ *(M1)*

$(d =)\ 11$ ***A1***

[2 marks]

(c) Find the period of the function in minutes. [3]

Markscheme

(time between high and low tide is) 6h15m **OR** 375 minutes ***(A1)***

multiplying by 2 ***(M1)***

750 minutes ***A1***

[3 marks]

(d) Find the value of b. [2]

Markscheme

EITHER

$\frac{360^\circ}{b} = 750$ ***(A1)***

OR

$7\ \cos(b \times 375) + 11 = 4$ ***(A1)***

THEN

$(b =)\ 0.48$ ***A1***

Note: Award ***A1A0*** for an answer of $\frac{2\pi}{750}$ $\left(= \frac{\pi}{375} = 0.00837758\ldots\right)$.

[2 marks]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\mathrm{m}$.

(e) Find the latest time before $12:00$, to the nearest minute, that Naomi can enter the harbour. [4]

Markscheme

equating their cos function to 6 **OR** graphing their cos function and 6 ***(M1)***

$7\cos(0.48t) + 11 = 6$

$\Rightarrow t = 282.468\ldots$ (minutes) ***(A1)***

$= 4.70780\ldots$ (hr) **OR** 4hr 42 mins (4hr $42.4681\ldots$ mins) ***(A1)***

so the time is $10:42$ ***A1***

[4 marks]

(f) Find the length of time (in minutes) between $06:00$ and $15:00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

Markscheme

next solution is $t = 467.531\ldots$ **(A1)**

$467.531\ldots - 282.468\ldots$

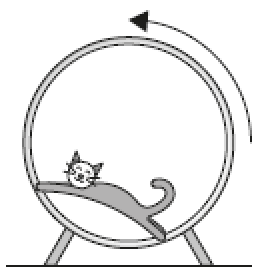
185 (mins) $(185.063\ldots)$ ***A1***

Note: Accept an (unsupported) answer of 186 (from correct 3 sf values for t)

[2 marks]

3. [Maximum mark: 6] 22N.1.SL.TZ0.12

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, h cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where t is the time in seconds and $a,\ b,\ c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of 78 cm.

(a) Write down the value of c. [1]

Markscheme

78 *A1*

[1 mark]

When $t = 4$, point P first reaches its maximum height of 143 cm.

(b.i) Find the value of a. [1]

Markscheme

65 *A1*

[1 mark]

(b.ii) Find the value of b. [2]

Markscheme

EITHER

(period =) 16 (could be seen on sketch) ***(M1)***

$b = \frac{2\pi}{16}$ **OR** $b = \frac{360^\circ}{16}$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^\circ$

A1

OR

$143 = 65\sin(4b) + 78$ ***(M1)***

$(\sin(4b) = 1)$

$(4b = \frac{\pi}{2}$ **OR** $4b = 90^\circ)$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^\circ$

A1

[2 marks]

(c) Write down the minimum height of point P. [1]

Markscheme

13 ***A1***

Note: Apply follow through marking only if their final answer is positive.

[1 mark]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b. [1]

Markscheme

$(b =)\ 0.196\ \left(0.196349\ldots,\ \frac{\pi}{16}\right)$ **OR** $(b =) 11.3^{\circ}\ \ (11.25^{\circ})$

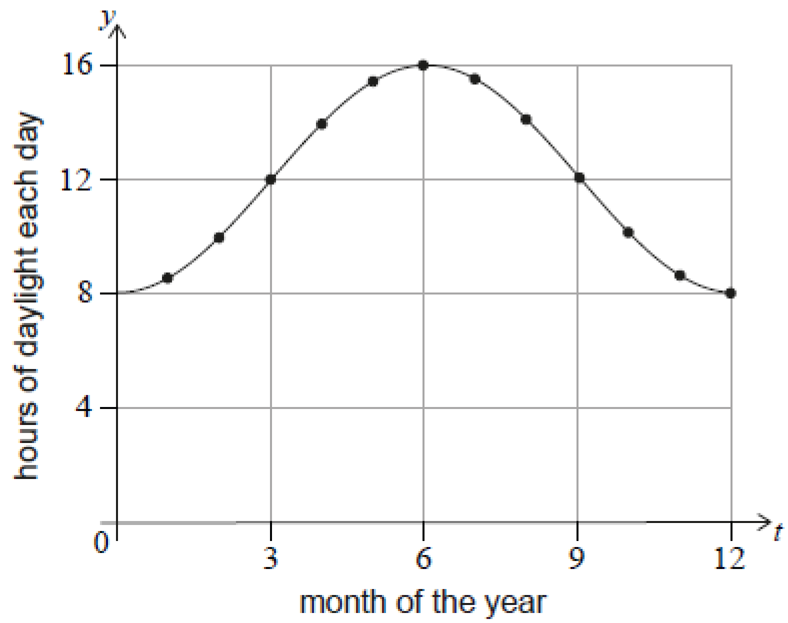
A1

[1 mark]

4. [Maximum mark: 15] 22M.2.SL.TZ1.1

Boris recorded the number of daylight hours on the first day of each month in a northern hemisphere town.

This data was plotted onto a scatter diagram. The points were then joined by a smooth curve, with minimum point $(0,\ 8)$ and maximum point $(6,\ 16)$ as shown in the following diagram.



Let the curve in the diagram be $y = f(t)$, where t is the time, measured in months, since Boris first recorded these values.

Boris thinks that $f(t)$ might be modelled by a quadratic function.

(a) Write down one reason why a quadratic function would not be a good model for the number of hours of daylight per day, across a number of years. [1]

Markscheme

EITHER

annual cycle for daylight length ***R1***

OR

there is a minimum length for daylight (cannot be negative) ***R1***

OR

a quadratic could not have a maximum and a minimum or equivalent ***R1***

Note: Do not accept "Paula's model is better".

[1 mark]

Paula thinks that a better model is $f(t) = a\ \cos(bt) + d, t \geq 0$, for specific values of $a,\ b$ and d.

For Paula's model, use the diagram to write down

(b.i) the amplitude. [1]

Markscheme

4 ***A1***

[1 mark]

(b.ii) the period. [1]

Markscheme

12 ***A1***

[1 mark]

(b.iii) the equation of the principal axis. [2]

Markscheme

$y = 12$ ***A1A1***

Note: Award ***A1*** "$y =$ (a constant)" and ***A1*** for that constant being 12.

[2 marks]

(c) Hence or otherwise find the equation of this model in the form:

$$f(t) = a\ \cos(bt) + d$$

[3]

Markscheme

$f(t) = -4\ \cos(30t) + 12$ **OR** $f(t) = -4\ \cos(-30t) + 12$ ***A1A1A1***

Note: Award ***A1*** for $b = 30$ (or $b = -30$), ***A1*** for $a = -4$, and ***A1*** for $d = 12$. Award at most ***A1A1A0*** if extra terms are seen or form is incorrect. Award at most ***A1A1A0*** if x is used instead of t.

[3 marks]

(d) For the first year of the model, find the length of time when there are more than 10 hours and 30 minutes of daylight per day. [4]

Markscheme

$10.5 = -4\cos(30t) + 12$ ***(M1)***

EITHER

$t_1 = 2.26585\ldots,\ \ t_2 = 9.73414\ldots$ ***(A1)(A1)***

OR

$t_1 = \frac{1}{30}\cos^{-1}\frac{3}{8}$ ***(A1)***

$t_2 = 12 - t_1$ ***(A1)***

THEN

$9.73414\ldots - 2.26585\ldots$

$7.47\quad(7.46828\ldots)$ months $(0.622356\ldots$ years)   ***A1***

Note: Award ***M1A1A1A0*** for an unsupported answer of 7.46. If there is only one intersection point, award ***M1A1A0A0***.

[4 marks]

The true maximum number of daylight hours was 16 hours and 14 minutes.

(e) Calculate the percentage error in the maximum number of daylight hours Boris recorded in the diagram. [3]

Markscheme

$$\left|\frac{16-\left(16+\frac{14}{60}\right)}{16+\frac{14}{60}}\right| \times 100\%$$ ***(M1)(M1)***

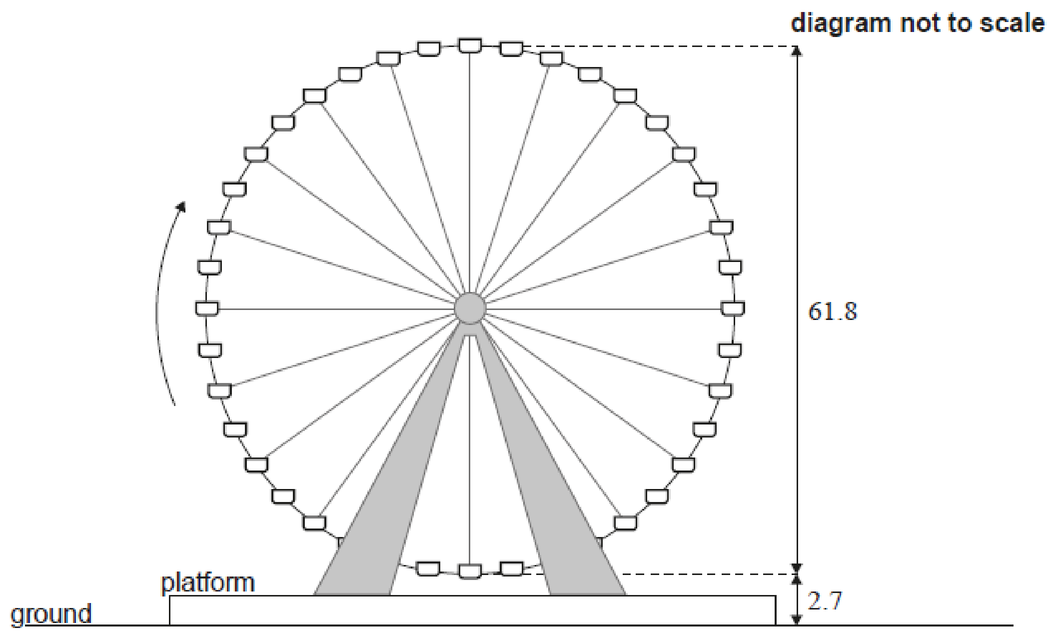
Note: Award ***M1*** for correct values and absolute value signs, ***M1*** for $\times 100$.

$$= 1.44\% \quad (1.43737\ldots\%)$$ ***A1***

[3 marks]

5. [Maximum mark: 17] 22M.2.SL.TZ2.4

The Texas Star is a Ferris wheel at the state fair in Dallas. The Ferris wheel has a diameter of 61.8 m. To begin the ride, a passenger gets into a chair at the lowest point on the wheel, which is 2.7 m above the ground, as shown in the following diagram. A ride consists of multiple revolutions, and the Ferris wheel makes 1.5 revolutions per minute.



The height of a chair above the ground, h, measured in metres, during a ride on the Ferris wheel can be modelled by the function $h(t) = -a\cos(bt) + d$, where t is the time, in seconds, since a passenger began their ride.

Calculate the value of

(a.i) a. [2]

Markscheme

an attempt to find the amplitude ***(M1)***

$\frac{61.8}{2}$ **OR** $\frac{64.5-2.7}{2}$

$(a =)\ 30.9\,\mathrm{m}$ ***A1***

Note: Accept an answer of $(a =)\ -30.9\,\mathrm{m}$.

[2 marks]

(a.ii) b. [2]

Markscheme

(period $= \frac{60}{1.5} =$) $40\,(\mathrm{s})$ ***(A1)***

($(b =)\ \frac{360^\circ}{40}$)

$(b =)\ 9$ ***A1***

Note: Accept an answer of $(b =)\ -9$.

[2 marks]

(a.iii) d. [2]

Markscheme

attempt to find d ***(M1)***

$(d =)\ 30.9 + 2.7$ **OR** $\frac{64.5+2.7}{2}$

$(d =)\ 33.6\,\mathrm{m}$ ***A1***

[2 marks]

A ride on the Ferris wheel lasts for 12 minutes in total.

(b) Calculate the number of revolutions of the Ferris wheel per ride. [2]

Markscheme

12×1.5 **OR** $\frac{12 \times 60}{40}$ ***(M1)***

18 (revolutions per ride) ***A1***

[2 marks]

For exactly one ride on the Ferris wheel, suggest

(c.i) an appropriate domain for $h(t)$. [1]

Markscheme

$0 \leq t \leq 720$ ***A1***

[1 mark]

(c.ii) an appropriate range for $h(t)$. [2]

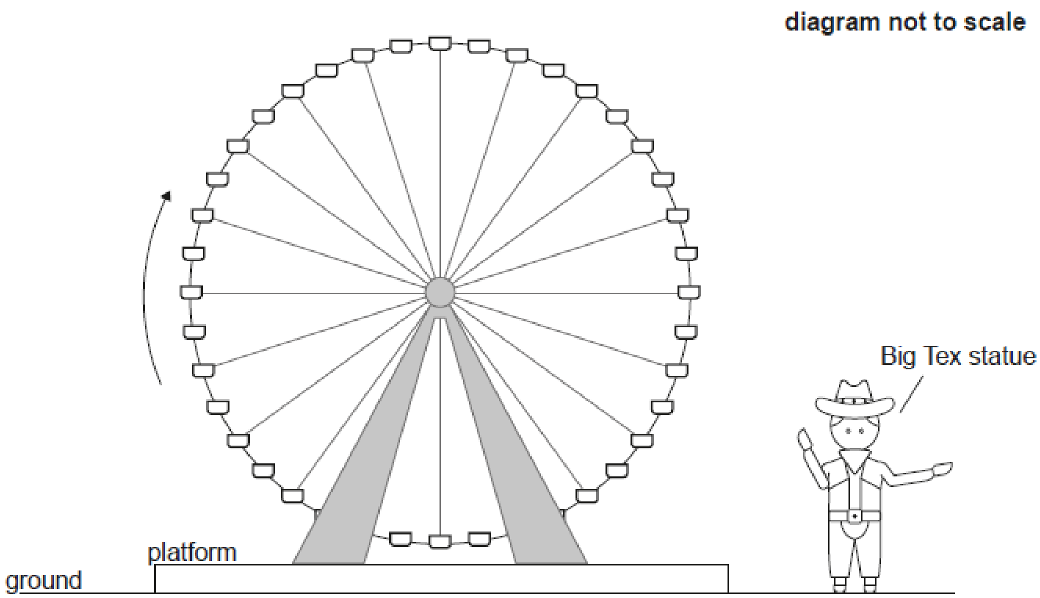
Markscheme

$2.7 \leq h \leq 64.5$ ***A1A1***

Note: Award ***A1*** for correct endpoints of domain and ***A1*** for correct endpoints of range. Award ***A1*** for correct direction of both inequalities.

[2 marks]

Big Tex is a 16.7 metre-tall cowboy statue that stands on the horizontal ground next to the Ferris wheel.



[Source: Aline Escobar., n.d. Cowboy. [image online] Available at: https://thenounproject.com/search/?q=cowboy&i=1080130

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(d) By considering the graph of $h(t)$, determine the length of time during one revolution of the Ferris wheel for which the chair is higher than the cowboy statue. [3]

Markscheme

graph of $h(t)$ and $y = 16.7$ **OR** $h(t) = 16.7$ ***(M1)***

$6.31596\ldots$ and $33.6840\ldots$ ***(A1)***

$27.4\,(\mathrm{s})\quad(27.3680\ldots)$ ***A1***

[3 marks]

There is a plan to relocate the Texas Star Ferris wheel onto a taller platform which will increase the maximum height of the Ferris wheel to $65.2\,\mathrm{m}$. This will change the value of one parameter, a, b or d, found in part (a).

(e.i) Identify which parameter will change. [1]

Markscheme

d ***A1***

[1 mark]

(e.ii) Find the new value of the parameter identified in part (e)(i). [2]

Markscheme

EITHER

$d + 30.9 = 65.2$ ***(A1)***

OR

$65.2 - (61.8 + 2.7) = 0.7$ ***(A1)***

OR

3.4 (new platform height) ***(A1)***

THEN

$(d =)\ 34.3\,\text{m}$ ***A1***

[2 marks]