Name:
Result:

1. *(7 points)*

Consider a complex number $z = 4 + 3i$.

(a) Find the modulus and argument of z. [2]

Let $w = 2\text{cis}\frac{\pi}{6}$.

(b) Describe the transformations that transform z into wz. [2]

Let A and B be the points on the Argand diagram that correspond to the complex numbers z and w^2z.

(c) Find the area of the triangle OAB, where O is the origin. [3]

(a) $|z| = 5$ and $\arg(z) = \arctan(\frac{3}{4}) = 0.644$

(b) Dilation (stretch) with scale factor of 2 and counter-clockwise rotation by $\frac{\pi}{6}$.

(c) We have $|OA| = 5, |OB| = 20$ (because OA was stretched by a factor of 2 twice) and $\angle AOB = \frac{\pi}{3}$ (because of the rotation). So the area is:

$$A = \frac{1}{2} \times 5 \times 20 \times \sin\left(\frac{\pi}{3}\right) = 25\sqrt{3}$$

2. *(10 points)*

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08\cos(0.0165t + 0.413) + 4.94$$

where t is the day of the year in 2021 (for example $t = 2$ represents 2 January 2021).

(a) State the earliest time of sunrise. [1]

$-1.08 + 4.94 = 3.86$, which gives 3:52

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15\cos(0.0165t - 2.97) + 18.9$$

(b) State the latest time of sunset. [1]

$1.15 + 18.9 = 20.05$, which gives 20:03

Let D be the number of daylight hours in Taipei during 2021.

(c) Explain, why the largest number of daylight hours is not necessarily your answer to part (a) subtracted from your answer to part (b). [1]

The earliest sunrise and latest sunset do not necessarily occur on the same day.

D can be expressed in the form:

$$D = a\cos(bt + c) + d$$

(d) State the value of b. $b = 0.0165$ [1]

(e) Find the values of a, c and d. [4]

We have $D = S - R$.

$$\begin{aligned} D &= 1.15\cos(0.0165t - 2.97) + 18.9 - 1.08\cos(0.0165t + 0.413) - 4.94 = \\ &= 1.15\cos(0.0165t - 2.97) - 1.08\cos(0.0165t + 0.413) + 13.96 = \\ &= \mathrm{Re}(1.15e^{i(0.0165t-2.97)} - 1.08e^{i(0.0165t+0.413)}) + 13.96 = \\ &= \mathrm{Re}(e^{0.0165ti}(1.15e^{-2.97i} - 1.08e^{0.413i})) + 13.96 = \\ &= \mathrm{Re}(2.21e^{i(0.0165t-2.85)}) + 13.96 = 2.21\cos(0.0165t - 2.85) + 13.96 \end{aligned}$$

(f) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. 16.2 on day 173 [2]

3. *(9 points)*

Let $z = 7 + i$.

(a) Plot the point z on the Argand diagram. [1]

Consider the line l given by the equation $y = \frac{1}{2}x$, where x denotes the real part and y the imaginary part.

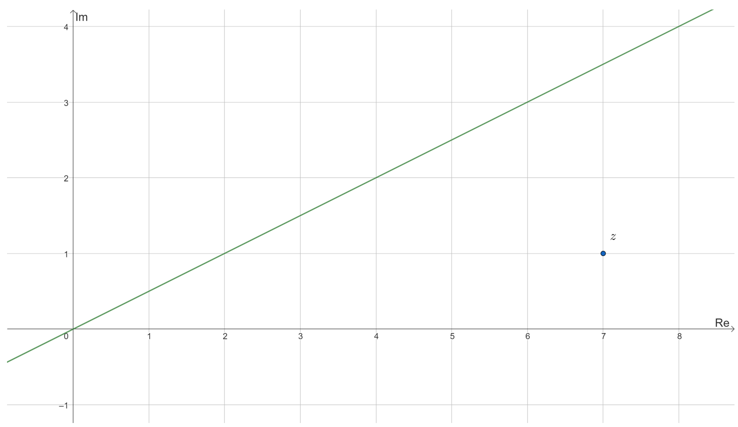
(b) Plot the line on the same Argand diagram. [1]

(c) Find the equation of the line perpendicular to l and passing through the point corresponding to z. [2]

(d) Find the complex number w which is the intersection of l and the line found in (c). [2]

(e) Hence, or otherwise, find z' the reflection of z in line l. [3]

Parts (a) and (b):



(c) The gradient must be -2, so the line is $y - 1 = -2(x - 7)$ or $y = -2x + 15$.
(d) We solve the system of equations:

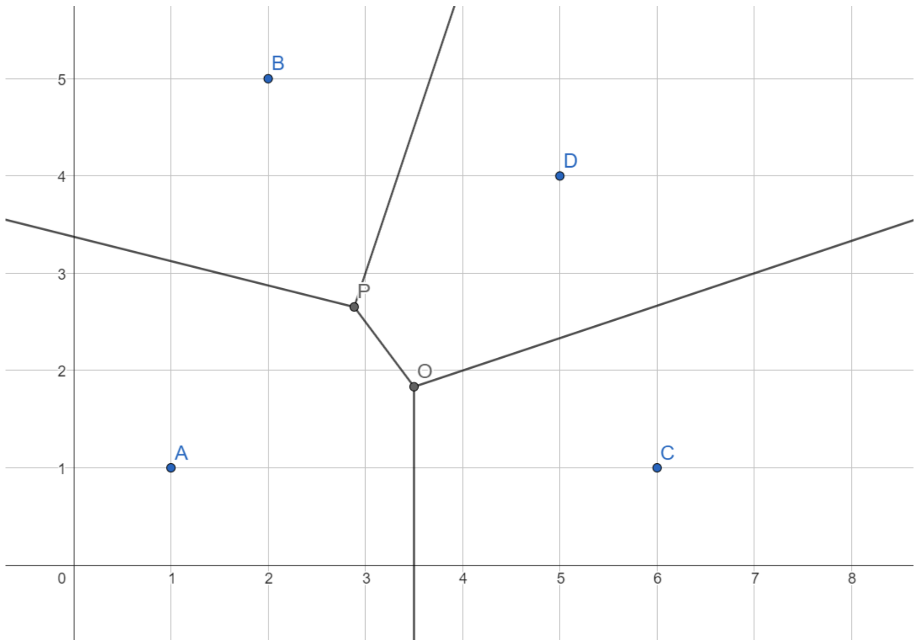
$$\begin{cases} y = \frac{1}{2}x \\ y = -2x + 15 \end{cases}$$

and we get $x = 6, y = 3$, so $w = 6 + 3i$.

(e) We can use vectors (the vector from $(7, 1)$ to $(6, 3)$ is $\binom{-1}{2}$) and we get that the point is $(5, 5)$, so $z' = 5 + 5i$.

4. *(9 points)*

The Voronoi diagram is constructed for sites $A(1,1), B(2,5), C(6,1)$ and $D(5,4)$.



The snowfall on 22 November 2023 is measured at these sites and the results are displayed below:

Site	A	B	C	D
Snowfall $[mm]$	11	12	5	7

(a) Use nearest neighbour interpolation to estimate the snowfall at the point $(4,5)$. [1]

(b) State the equation of the perpendicular bisector of $[AC]$ [1]

(c) Find equation of perpendicular bisector of $[CD]$. [2]

(d) Hence, or otherwise, find the coordinates of the vertex Q. [2]

The coordinates of vertex P are $(2.88, 2.65)$ correct to 3 significant figures. A new site is to be added to the diagram within the quadrilateral $ABCD$ and as far as possible from all other sites.

(e) State the location of the new site. Justify your answer. [3]

(a) The nearest site is D, so 7 mm.

(b) $x = 3.5$

(c) The vector $\overrightarrow{DC} = \binom{1}{-3}$, the midpoint is $M_{CD} = (5.5, 2.5)$, so the equation is

$$x - 3y + 2 = 0$$

(d) We solve the system of equations (but one equation is just $x = 3.5$) and we get that $x = 3.5$ and $y = 1.8(3)$. So $Q = (3.5, 1.8(3))$.

(e) We need to compare the distance from P and from Q to the nearest site (for example A). We get:

$$|AP| = \sqrt{1.88^2 + 1.65^2} \approx 2.50$$

$$|AQ| = \sqrt{2.5^2 + 0.8(3)^2} \approx 2.64$$

We have $|AQ| > |AP|$, so the new site should be located at Q.