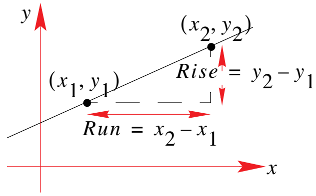
Properties of straight lines

1. Gradient of a line

The gradient, m, of the line through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{\text{Rise}}{\text{Run}}$



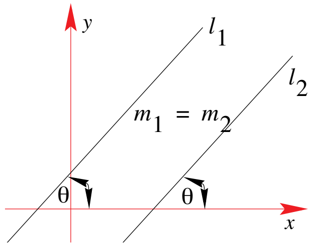
From this we can obtain the point–gradient form of a line.
That is, if (x, y) is any point on a straight line having a gradient m, and (x_1, y_1) is another fixed point on that line then the equation of that line is given by

$$y - y_1 = m(x - x_1)$$

2. Parallel lines

The straight line l_1 with gradient m_1 is parallel to the straight line l_2 with gradient m_2 if and only if $m_1 = m_2$.

That is, $l_1 \mathbin{/\!/} l_2$ iff $m_1 = m_2$

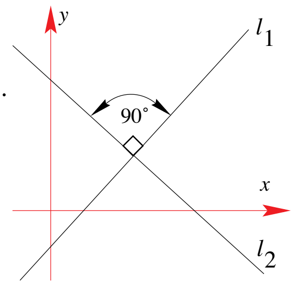


Notice that if the two lines are parallel, they also make equal angles with the x–axis.

3. Perpendicular lines

The straight line l_1 with gradient m_1 is perpendicular to the straight line l_2 with gradient m_2 if and only if $m_1 \times m_2 = -1$.

That is, $l_1 \perp l_2$ iff $m_1 \times m_2 = -1$ or $m_1 = -\frac{1}{m_2}$.



EXAMPLE 2.16

Find the equation of the line that passes through the point (–1,3) and is parallel to the line with equation $2x - y + 7 = 0$.

Solution

The gradient of the line $2x - y + 7 = 0$ is found by rearranging to the form $y = mx + c$ to get: $y = 2x + 7$. The gradient is 2 and so all the lines parallel to this will also have gradient 2. The equation of the required line is $y = 2x + c$. The value of the constant c can be found by using the fact that the line passes through the point (–1,3).
That is, $y = 2x + c \therefore 3 = 2 \times -1 + c \Leftrightarrow c = 5$
Therefore the equation of the straight line is $y = 2x + 5$.

EXAMPLE 2.17

Find the equation of the line which passes through the point (–1,4) and which is perpendicular to the line with equation $2x + 5y + 2 = 0$.

Solution

The gradient form of $2x+5y+2 = 0$ is $5y = -2x-2 \Rightarrow y = -\frac{2}{5}x-\frac{2}{5}$.

So the gradient is $-\frac{2}{5}$. The gradient of all lines perpendicular to this line is found using the fact that the product of the gradients of perpendicular lines is –1: $\left(-\frac{2}{5}\right)m = -1 \Rightarrow m = \frac{5}{2} = 2.5$.

Then, the equation of the line is $y = \frac{5}{2}x+c$. The constant c is found in the same way as the previous example: Using the point (–1, 4) we have $4 = \frac{5}{2}\times -1+c \Leftrightarrow c = 6.5$.

Therefore the equation of the straight line is $y = \frac{5}{2}x+6\frac{1}{2}$.

EXERCISES 2.3.1

1. Sketch the graph of the following straight lines.

(i) $y = x+1$	(ii) $f(x) = x-2$	(iii) $y = 2x-3$
(iv) $f(x) = 2-3x$	(v) $y = \frac{x+1}{2}$	(vi) $f(x) = 3+4x$
(vii) $x+f(x) = 3$	(viii) $x+2y = 4$	(ix) $x-3y = 6$
(x) $\frac{x}{2}+\frac{y}{5} = 1$	(xi) $x-\frac{y}{3} = 1$	(xii) $\frac{2x}{5}-3y = 2$
(xiii) $x+\frac{5y}{4} = -1$	(xiv) $\frac{4+t}{2} = q$	(xv) $x+4y = 2-x$

2. Find the gradient of the line joining the points
(a) (3, 2) and (5, 6) (b) (4, 5) and (6, 11) (c) (–1, 3) and (2, 8)

3. Use the gradient–point method to find the equation of the straight line if
(a) it passes through the point (1, 1) and has a gradient of 2.
(b) it passes through the point (–2, 3) and has a gradient of 3.
(c) it passes through the point (3, –4) and has a gradient of –1.

4. Find the gradient of the straight line that is perpendicular to the straight line with gradient equal to
(a) 2. (b) – 3. (c) $-\frac{2}{3}$. (d) $\frac{5}{4}$

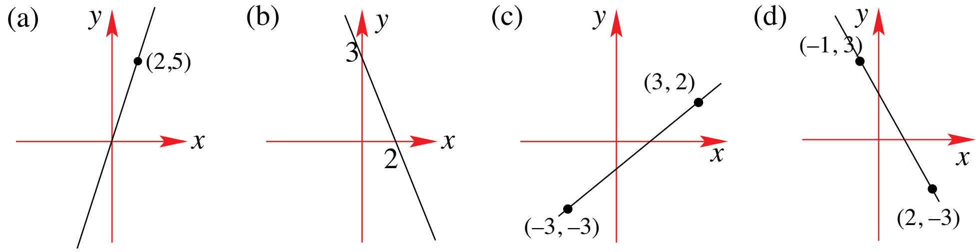
5. Find the equation of the straight line that passes through the origin and the point (2,4).

6. Find the equation of the straight line that passes through the points (–1,2) and (0,1).

7. A straight line passes through the point (4,3) and is perpendicular to the line joining the points (–1,3) and (1,–1). Find the equation of this line.

8. The lines $px + 4y - 2 = 0$ and $2x - y + p = 0$ are perpendicular. Find the value of p.

9. Find equations for each of the following lines.



10. Sketch the graph of the following functions.

(a) $f(x) = ax - b, a < 0, b > 0$

(b) $f(x) = a^2x + b, b < 0, a \neq 0$

(c) $f(x) = \frac{a}{a+1}x - a, a > 0$

(d) $f(x) = 2a + \frac{1}{a}x, a > 0$

2.3.2 SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Pairs of simultaneous equations in two unknowns may be solved in two ways, either algebraically or graphically. To solve means to find where the two straight lines intersect once they have been sketched. So, we are looking for the point of intersection.

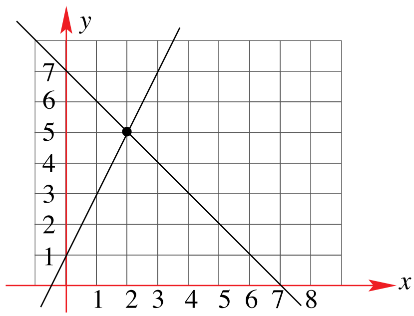
Method 1: Graphical

EXAMPLE 2.18

Solve the system of linear equations $y = -x + 7$ and $y = 2x + 1$.

Solution

We sketch both lines on the same set of axes:



Reading off the grid we can see that the straight lines meet at the point with coordinates (2, 5).

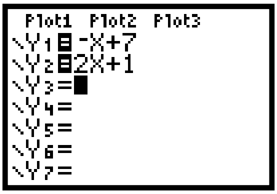
So, the solution to the given system of equations is $x = 2$ and $y = 5$.

The graphical approach has a disadvantage. Sometimes it can only provide an approximate answer. This depends on the accuracy of the sketch or simply on the equations, for example, sketching the pair of straight lines with equations $y = \sqrt{2}x - 1$ and $y = -x + \sqrt{3}$ can only result in an approximate solution.

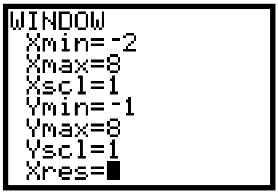
Then there is the graphics calculator. There are a number of ways that the graphics calculator can be used. Using the TI–83 we can make use of the **TRACE** function or the **intersection** option under the **CALC** menu. We solve the previous problem using the intersection option from the CALC menu.

Step 1: Enter both equations in the required form, i.e., $y =$ . . .
Step 2: Choose an appropriate window setting, in this case we have [–2,8] by [–1,8].
Step 3: Sketch the straight lines using the **GRAPH** key.
Step 4: Call up the **CALC** menu (i.e., press **2nd TRACE**) and choose option **5: intersect**.
Step 5:
5.1 Move the cursor to where the lines intersect and press **ENTER** – this confirms that you have selected your first equation.
5.2 Press **ENTER** again, this confirms that you are using the second equation.
5.3 Because you have already placed your cursor near the point of intersection, when prompted to Guess? simple press **ENTER**.

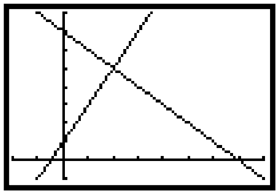
Step 1:



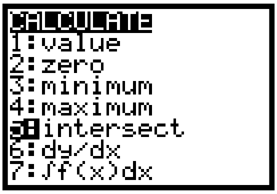
Step 2:



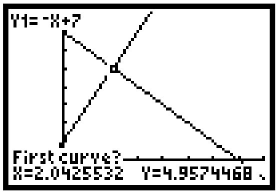
Step 3:



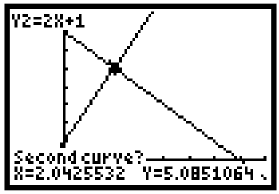
Step 4:



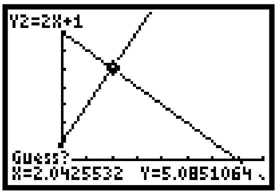
Step 5.1:

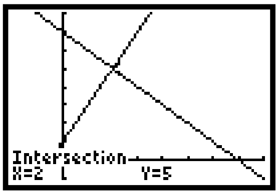


Step 5.2:



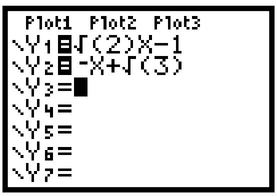
Step 5.3:

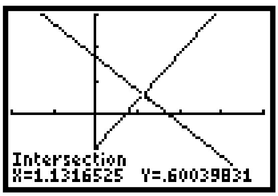




The final screen shows the solution as $x = 2$ and $y = 5$.

This particular example worked out rather neatly, however, the solution of the pairs of equations $y = \sqrt{2}x - 1$ and $y = -x + \sqrt{3}$ would produce the following result:





i.e., we only obtain an approximate solution!

The exact solution is in fact $x = \sqrt{6} + \sqrt{2} - \sqrt{3} - 1$, $y = 2\sqrt{3} + 1 - \sqrt{6} - \sqrt{2}$.

Of course, depending on the application, an approximate solution might suffice. However, at this stage we are interested in the mathematical process. Because we cannot always obtain an exact answer using a graphical means we need to consider an algebraic approach.

Method 2: Algebraic

There are two possible approaches when dealing with simultaneous equations algebraically. They are the process of
a. Elimination
b. Substitution

Elimination Method

The **key step** in using the elimination method is to obtain, for one of the variables (in both equations), coefficients that are the same (or only differ in sign). Then:

if the coefficients are the same, you subtract one equation from the other – this will **eliminate** one of the variables – leaving you with only one unknown.

However,

if the coefficients only differ in sign, you add the two equations – this will **eliminate** one of the variables – leaving you with only one unknown.

EXAMPLE 2.19

Use the elimination method to solve
$$\begin{aligned} x - 2y &= -7 \\ 2x + 3y &= 0 \end{aligned}$$

Solution

As it is easier to add than subtract, we try to eliminate the variable which differs in sign. In this case the variable 'y' is appropriate. However, the coefficients still need to be manipulated. We label the equations as follows:

$$x - 2y = -7 \quad -(1)$$
$$2x + 3y = 0 \quad -(2)$$
$3\times(1)$: $$3x - 6y = -21 \quad -(3)$$
$2\times(2)$: $$4x + 6y = 0 \quad -(4)$$
Adding $(3) + (4)$: $$7x + 0 = -21$$
$$\Leftrightarrow x = -3$$

Substituting into (1) we can now obtain the y–value: $-3 - 2y = -7 \Leftrightarrow -2y = -4 \Leftrightarrow y = 2$.
Therefore, the solution is $x = -3$, $y = 2$.

Once you have found the solution, always check with one of the original equations.
Using equation (2) we have: L.H.S $= 2 \times -3 + 3 \times 2 = 0 =$ R.H.S.

Note that we could also have multiplied equation (1) by 2 and then subtracted the result from equation (2). Either way, we have the same answer.

Substitution Method

The substitution method relies on making one of the variables the subject of one of the equations. Then we substitute this equation for its counterpart in the other equation. This will then produce a new equation that involves only one unknown. We can solve for this unknown and then substitute its value back into the first equation. This will then provide a solution pair.

EXAMPLE 2.20

Use the substitution method to solve $\begin{aligned} 5x - y &= 4 \\ x + 3y &= 4 \end{aligned}$

Solution

Label the equations as follows: $5x - y = 4 \quad -(1)$

$x + 3y = 4 \quad -(2)$

From equation (1) we have that $y = 5x - 4 \quad -(3)$

Substituting (3) into (2) we have: $x + 3(5x - 4) = 4$

$\Leftrightarrow 16x - 12 = 4$

$\Leftrightarrow 16x = 16$

$\Leftrightarrow x = 1$

Substituting $x = 1$ into equation (3) we have: $y = 5 \times 1 - 4 = 1$.

Therefore, the solution is given by $x = 1$ and $y = 1$.

Check: Using equation (2) we have: L.H.S $= 1 + 3 \times 1 = 4 =$ R.H.S

Not all simultaneous equations have unique solutions. Some pairs of equations have no solutions while others have infinite solution sets. You will need to be able to recognise the 'problem' in the processes of both algebraic and graphical solutions when dealing with such equations.

The following examples illustrate these possibilities:

EXAMPLE 2.21

Solve: (a) $\begin{aligned} 2x + 6y &= 8 \\ 3x + 9y &= 12 \end{aligned}$ (b) $\begin{aligned} 2x + 6y &= 8 \\ 3x + 9y &= 15 \end{aligned}$

Solution

(a) Algebraic solution:

Label the equations as follows:

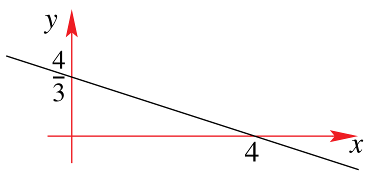
$2x + 6y = 8 \quad -(1)$

$3x + 9y = 12 \quad -(2)$

$3\times(1)$: $6x + 18y = 24 \quad -(3)$

$2\times(2)$: $6x + 18y = 24 \quad -(4)$

Graphical Solution:



In this case, we have the same equation. That is, the straight lines are **coincident**.

If we were to 'blindly' continue with the solution process, we would have:

$3\times(1) - 2\times(2)$: $0 = 0!$

The algebraic method produces an equation that is always true, i.e., zero will always equal zero. This means that any pair of numbers that satisfy either equation will satisfy both and are, therefore, solutions to the problem. Examples of solutions are: $x = 4, y = 0, x = 1, y = 1$ & $x = 7$, $y = -1$. In this case we say that there is an **infinite** number of solutions.

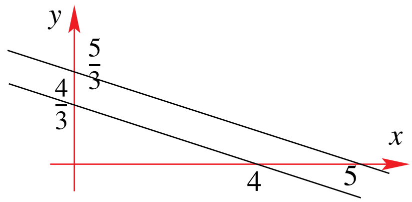
Graphically, the two equations produce the same line. The coordinates of any point on this line will be solutions to both equations.

(a) Algebraic solution:

Label the equations as follows:

$$\begin{aligned} 2x + 6y &= 8 \quad -(1) \\ 3x + 9y &= 15 \quad -(2) \\ 3\times(1): \quad 6x + 18y &= 24 \quad -(3) \\ 2\times(2): \quad 6x + 18y &= 30 \quad -(4) \\ (4)-(3): \quad 0 &= 6 \end{aligned}$$

Graphical Solution:



The algebraic method produces an equation that is never true. This means that there are no solutions to the equations.

Graphically, the two lines are parallel and produce no points of intersection.

There is a **matrix** method for solving systems of simultaneous equations that will be discussed in Chapter 25. This method is particularly useful when using graphics calculators which can perform the matrix arithmetic necessary to solve simultaneous equations.

EXERCISES 2.3.2

1. Solve these simultaneous equations, giving exact answers.

(i) $\begin{aligned} 3x - 2y &= -1 \\ 5x + 2y &= 9 \end{aligned}$ (ii) $\begin{aligned} 3x + 5y &= 34 \\ 3x + 7y &= 44 \end{aligned}$ (iii) $\begin{aligned} 2x + 4y &= 6 \\ 4x - 3y &= -10 \end{aligned}$

(iv) $\begin{aligned} 3x + 2y &= 2 \\ 2x - 6y &= -6 \end{aligned}$ (v) $\begin{aligned} 5x + 4y &= -22 \\ 3x - y &= -3 \end{aligned}$ (vi) $\begin{aligned} 5x - 9y &= -34 \\ 2x + 3y &= -7 \end{aligned}$

2. Solve these simultaneous equations, giving fractional answers where appropriate.

(i) $\begin{aligned} 3x - y &= 2 \\ 5x + 2y &= 9 \end{aligned}$ (ii) $\begin{aligned} 4x + 2y &= 3 \\ x - 3y &= 0 \end{aligned}$ (iii) $\begin{aligned} -3x + y &= 0 \\ 2x - 4y &= 0 \end{aligned}$

(iv) $\begin{aligned} \frac{x}{2} - 3y &= 4 \\ 4x + \frac{3y}{2} &= -1 \end{aligned}$ (v) $\begin{aligned} 5x + \frac{2y}{3} &= -4 \\ 4x + y &= 2 \end{aligned}$ (vi) $\begin{aligned} \frac{3x}{5} - 4y &= \frac{1}{2} \\ x - 2y &= \frac{1}{3} \end{aligned}$

3. Find the values of m such that these equations have no solutions.

(i) $\begin{aligned} 3x - my &= 4 \\ x + y &= 12 \end{aligned}$ (ii) $\begin{aligned} 5x + y &= 12 \\ mx - y &= -2 \end{aligned}$ (iii) $\begin{aligned} 4x - 2y &= 12 \\ 3x + my &= 2 \end{aligned}$

4. Find the values of m and a such that these equations have infinite solution sets.

(i) $\begin{aligned} 4x + my &= a \\ 2x + y &= 4 \end{aligned}$ (ii) $\begin{aligned} 5x + 2y &= 12 \\ mx + 4y &= a \end{aligned}$ (iii) $\begin{aligned} 3x + my &= a \\ 2x - 4y &= 6 \end{aligned}$

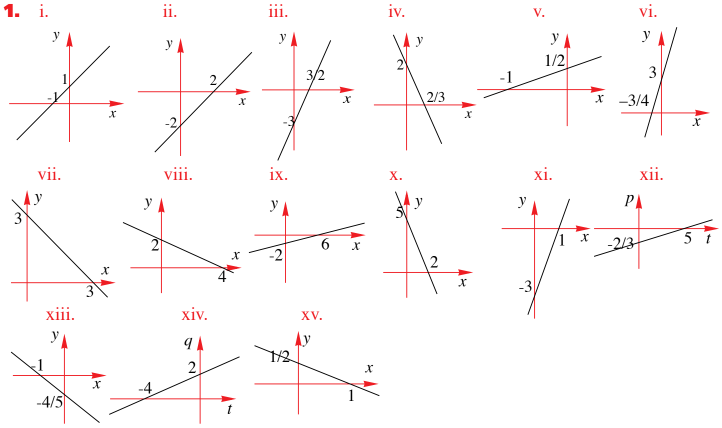
(h) $x < -\frac{5}{12} \cup x > \frac{1}{12}$ (i) $x \le -4 \cup x \ge 16$ **6.** $p < 3$ **7.** (a) $-\frac{2}{3} < x < 2$ (b) $-3 \le x \le 1$

(c) $0 < x < 2$ **8.** (a) $\frac{a}{1+a} < x < \frac{a}{1-a}$ (b) $\frac{-1}{1+a} < x < \frac{1}{1-a}$ (c) $\left]-\infty, \frac{-a^2}{a+1}\right] \cup \left[\frac{a^2}{a-1}, \infty\right[$

9. (a) $-\frac{4}{3} < x < \frac{4}{3}$ (b) $-\frac{3}{2} < x < \frac{3}{4}$

EXERCISE 2.3.1

1.

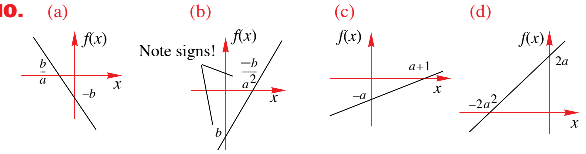


2. (a) 2 (b) 3 (c) $\frac{5}{3}$ **3.** (a) $y = 2x - 1$ (b) $y = 3x + 9$ (c) $y = -x - 1$ **4.** (a) $-\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{3}{2}$

(d) $-\frac{4}{5}$ **5.** $y = 2x$ **6.** $y = -x + 1$ **7.** $y = \frac{x+2}{2}$ **8.** 2 **9.** (a) $y = \frac{5}{2}x$ (b) $y = -\frac{3}{2}x + 3$

(c) $y = \frac{5}{6}x - \frac{1}{2}$ (d) $y = -2x + 1$

10. (a) (b) (c) (d)



EXERCISE 2.3.2

1. (i) $x = 1, y = 2$ (ii) $x = 3, y = 5$ (iii) $x = -1, y = 2$ (iv) $x = 0, y = 1$ (v) $x = -2, y = -3$

(vi) $x = -5, y = 1$ **2.** (i) $x = \frac{13}{11}, y = \frac{17}{11}$ (ii) $x = \frac{9}{14}, y = \frac{3}{14}$ (iii) $x = 0, y = 0$

(iv) $x = \frac{4}{17}, y = -\frac{22}{17}$ (v) $x = -\frac{16}{7}, y = \frac{78}{7}$ (vi) $x = \frac{5}{42}, y = -\frac{3}{28}$ **3.** (i) –3 (ii) –5 (iii) –1.5

4. (i) $m = 2, a = 8$ (ii) $m = 10, a = 24$ (iii) $m = -6, a = 9$.

5. (a) $x = 1, y = a - b$ (b) $x = -1, y = a + b$ (c) $x = \frac{1}{a}, y = 0$ (d) $x = b, y = 0$

(e) $x = \frac{a-b}{a+b}, y = \frac{a-b}{a+b}$ (f) $x = a, y = b - a^2$

EXERCISE 2.3.3

1. (a) $x = 4, y = -5, z = 1$ (b) $x = 0, y = 4, z = -2$ (c) $x = 10, y = -7, z = 2$

(d) $x = 1, y = 2, z = -2$ (e) $\varnothing$ (f) $x = 2t - 1, y = t, z = t$ (g) $x = 2, y = -1, z = 0$ (h) $\varnothing$

EXERCISE 2.4.1

1. (a) –5 (b) 4, 6 (c) –3, 0 (d) 1, 3 (e) –6, 3 (f) $-2, \frac{5}{3}$ (g) 2 (h) –3, 6 (i) –6, 1 (j) $0, \frac{3}{2}$

2. (a) –1 (b) –7, 5 (c) $-\frac{2}{5}, 3$ (d) –2, 1 (e) –3, 1 (f) 4, 5

3. (a) $-1 \pm \sqrt{6}$ (b) $3 \pm \sqrt{5}$ (c) $1 \pm \sqrt{5}$ (d) $\frac{-1 \pm \sqrt{33}}{8}$ (e) $\frac{9 \pm \sqrt{73}}{4}$ (f) $\frac{1 \pm \sqrt{85}}{6}$

4. (a) $\frac{3 \pm \sqrt{37}}{2}$ (b) $\frac{5 \pm \sqrt{33}}{2}$ (c) $\frac{3 \pm \sqrt{33}}{2}$ (d) $\frac{7 \pm \sqrt{57}}{2}$ (e) $\frac{-7 \pm \sqrt{65}}{2}$ (f) –4, 2 (g) $-1 \pm 2\sqrt{2}$

(h) $\frac{-5 \pm \sqrt{53}}{2}$ (i) $\frac{3 \pm \sqrt{37}}{2}$ (j) no real solutions (k) $4 \pm \sqrt{7}$ (l) no real solutions (m) $\frac{2 \pm \sqrt{13}}{2}$

(n) $\frac{3 \pm 2\sqrt{11}}{5}$ (o) $\frac{6 \pm \sqrt{31}}{5}$ (p) $\frac{6 \pm \sqrt{29}}{7}$ **5.** (a) $-2 < p < 2$ (b) $p = \pm 2$ (c) $p < -2$ or $p > 2$

6. (a) $m = 1$ (b) $m < 1$ (c) $m > 1$ **7.** (a) $m = \pm 2\sqrt{2}$ (b) $]-\infty, -2\sqrt{2}[\cup]2\sqrt{2}, \infty[$

(c) $]-2\sqrt{2}, 2\sqrt{2}[$ **8.** (a) $k = \pm 6\sqrt{2}$ (b) $]-\infty, -6\sqrt{2}[\cup]6\sqrt{2}, \infty[$ (c) $]-6\sqrt{2}, 6\sqrt{2}[$ **10.** 4

EXERCISE 2.4.2

1. Graphs are shown using the ZOOM4 viewing window: