Matrices - intro [146 marks]

1. [Maximum mark: 4] EXM.1.AHL.TZ0.2

If $A = \begin{pmatrix} 2p & 3 \\ -4p & p \end{pmatrix}$ and det $A = 14$, find the possible values of p. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2p^2 + 12p = 14$ ***(M1)(A1)***

$p^2 + 6p - 7 = 0$

$(p + 7)(p - 1) = 0$ ***(A1)***

$p = -7$ or $p = 1$ ***(A1)(C4)***

Note: Both answers are required for the final ***(A1)***.

[4 marks]

2. [Maximum mark: 4] EXM.1.AHL.TZ0.3

A and B are 2×2 matrices, where $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and

$BA = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find B [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$B = (BA)A^{-1}$ *(M1)*

$= -\frac{1}{4}\begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix}$ *(M1)*

$= -\frac{1}{4}\begin{pmatrix} -4 & -12 \\ -16 & -48 \end{pmatrix}$ *(A1)*

$= \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ *(A1)*

OR

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$ *(M1)*

$\Rightarrow \left.\begin{matrix} 5a + 2b = 11 \\ 2a \qquad = 2 \end{matrix}\right\}$

$\Rightarrow a = 1, b = 3$ *(A1)*

$\left.\begin{matrix} 5c + 2d = 44 \\ 2c \ = 8 \end{matrix}\right\}$

$\Rightarrow c = 4, d = 12$ *(A1)*

$B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ ***(A1) (C4)***

Note: Correct solution with inversion (ie *AB* instead of *BA*) earns FT marks, (maximum ***[3 marks]***).

[4 marks]

3. [Maximum mark: 6] EXM.1.AHL.TZ0.28

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} e^x & e^{-x} \\ 2 + e^x & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of x for which $\boldsymbol{A}$ is singular. [6]

Markscheme

finding $\det \boldsymbol{A} = e^x - e^{-x}(2 + e^x)$ or equivalent ***A1***

$\boldsymbol{A}$ is singular $\Rightarrow \det \boldsymbol{A} = 0$ ***(R1)***

$e^x - e^{-x}(2 + e^x) = 0$

$e^{2x} - e^x - 2 = 0$ ***A1***

solving for e^x ***(M1)***

$e^x > 0$ (or equivalent explanation) ***(R1)***

$e^x = 2$

$x = \ln 2$ (only) ***A1 N0***

[6 marks]

4. [Maximum mark: 5] EXM.1.AHL.TZ0.32

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find k. [5]

Markscheme

$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} 1+2k & 0 \\ 0 & 2k+1 \end{pmatrix}$ ***A2***

Note: Award ***A2*** for 4 correct, ***A1*** for 2 or 3 correct.

$1 + 2k = 0$ ***M1***

$k = -\frac{1}{2}$ ***A1***

[5 marks]

5. [Maximum mark: 5] EXM.1.AHL.TZ0.33

Given that $\boldsymbol{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and that $\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = 0$ find k. [5]

Markscheme

$\boldsymbol{M}^2 = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix}$ ***M1A1***

$\Rightarrow \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix} - \begin{pmatrix} 12 & -6 \\ -18 & 24 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(M1)***

$\Rightarrow \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(A1)***

$\Rightarrow k = 5$ ***A1***

[5 marks]

6. [Maximum mark: 6] EXM.1.AHL.TZ0.34

The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{X})$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$. [6]

Markscheme

For multiplying $(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2)$ ***M1***

$= \boldsymbol{I}^2 + \boldsymbol{IX} + \boldsymbol{IX}^2 - \boldsymbol{XI} - \boldsymbol{X}^2 - \boldsymbol{X}^3 = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2 - \boldsymbol{X} - \boldsymbol{X}^2 - \boldsymbol{X}^3$ ***(A1)(A1)***

$= \boldsymbol{I} - \boldsymbol{X}^3$ ***A1***

$= \boldsymbol{I}$ ***A1***

$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{B}$ ***(R1)***

$(\boldsymbol{I} - \boldsymbol{X})\ (\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2) = \boldsymbol{I} \Rightarrow (\boldsymbol{I} - \boldsymbol{X})^{-1} = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$ ***AG NO***

[5 marks]

7. [Maximum mark: 3] EXM.1.AHL.TZ0.44

Find the values of the real number k for which the determinant of the matrix $\begin{pmatrix} k-4 & 3 \\ -2 & k+1 \end{pmatrix}$ is equal to zero. [3]

Markscheme

$$\begin{vmatrix} k-4 & 3 \\ -2 & k+1 \end{vmatrix} = 0$$

$\Rightarrow (k-4)(k+1)+6=0$ ***(M1)***

$\Rightarrow k^2-3k+2=0$ ***(M1)***

$\Rightarrow (k-2)(k-1)=0$

$\Rightarrow k=2$ or $k=1$ ***(A1) (C3)***

[3 marks]

8. [Maximum mark: 3] EXM.1.AHL.TZ0.46

If $\boldsymbol{A} = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}$, find 2 values of x and y, given that ***AB*** = ***BA***. [3]

Markscheme

AB = $\begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 2x + 32 & xy + 16 \\ 24 & 4y + 8 \end{pmatrix}$ ***(A1)***

BA = $\begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix} \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2x + 4y & 2y + 8 \\ 8x + 16 & 40 \end{pmatrix}$ ***(A1)***

AB = ***BA*** ⇒ $8x + 16 = 24$ and $4y + 8 = 40$

This gives $x = 1$ and $y = 8$. ***(A1) (C3)***

[3 marks]

9. [Maximum mark: 6] EXM.1.AHL.TZ0.50

Given that $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, find $\boldsymbol{X}$ if $\boldsymbol{BX} = \boldsymbol{A} - \boldsymbol{AB}$.

[6]

Markscheme

METHOD 1

$\boldsymbol{A} - \boldsymbol{AB} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} 4 & -9 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{X} = \boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{AB}) = \boldsymbol{B}^{-1}\begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$ ***(M1)***

$= -\frac{1}{6}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$ ***(A1)***

$= \begin{pmatrix} -1 & 6 \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix}$ ***(A2) (C6)***

METHOD 2

Attempting to set up a matrix equation ***(M2)***

$\boldsymbol{X} = \boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{AB})$ ***(A2)***

$= \begin{pmatrix} -1 & 6 \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix}$ (from GDC) ***(A2) (C6)***

[6 marks]

10. [Maximum mark: 6] EXM.1.AHL.TZ0.7

Let $\begin{pmatrix} b & 3 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 9 & 5 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ a & 15 \end{pmatrix}$.

(a.i) Write down the value of a. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$a = 5$ ***A1 N1***

[1 mark]

(a.ii) Find the value of b. [2]

Markscheme

$b + 9 = 4$ ***(M1)***

$b = -5$ ***A1 N2***

[2 marks]

(b) Let $3\begin{pmatrix} -4 & 8 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 \\ q & -4 \end{pmatrix} = \begin{pmatrix} -22 & 24 \\ 9 & 23 \end{pmatrix}$.

Find the value of q. [3]

Markscheme

Comparing elements $3(2) - 5(q) = -9$ ***M1***

$q = 3$ ***A2 N2***

[3 marks]

11. [Maximum mark: 6] EXM.1.AHL.TZ0.35

(a) Write down the inverse of the matrix

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$$

[2]

Markscheme

$$A^{-1} = \begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$$ ***A2 N2***

[2 marks]

(b) **Hence**, find the point of intersection of the three planes.

$$x - 3y + z = 1$$
$$2x + 2y - z = 2$$
$$x - 5y + 3z = 3$$

[3]

Markscheme

For attempting to calculate $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ***M1***

$x = 1.2,\ y = 0.6,\ z = 1.6$ (so the point is (1.2, 0.6, 1.6)) ***A2 N2***

[3 marks]

(c) A fourth plane with equation $x + y + z = d$ passes through the point of intersection. Find the value of d. [1]

Markscheme

(1.2, 0.6, 1.6) lies on $x + y + z = d$

$\therefore d = 3.4$ ***A1 N1***

[1 mark]

12. [Maximum mark: 4] EXM.1.AHL.TZ0.43

(a) Find the values of a and b given that the matrix

$A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix

$B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$. [2]

Markscheme

$AB = I$

$(AB)_{11} = 1 \Rightarrow a - 12 + 6 = 1$, giving $a = 7$ ***(A1)(C1)***

$(AB)_{22} = 1 \Rightarrow -16 + 5b + 7 = 1$, giving $b = 2$ ***(A1)(C1)***

[2 marks]

(b) For the values of a and b found in part (a), solve the system of linear equations

$$x + 2y - 2z = 5$$
$$3x + by + z = 0$$
$$-x + y - 3z = a - 1.$$

[2]

Markscheme

the system is $BX = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Then, $X = A\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$. ***(M1)***

Thus $x = -1, y = 2, z = -1$ ***(A1) (C2)***

[2 marks]

13. [Maximum mark: 7] EXM.1.AHL.TZ0.16

(a) Find a relationship between a and b if the matrices $M = \begin{pmatrix} 1 & a \\ 2 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & b \\ 2 & 3 \end{pmatrix}$ commute under matrix multiplication. [4]

Markscheme

$$\begin{pmatrix} 1 & a \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & b \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1+2a & b+3a \\ 8 & 2b+9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & a \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1+2b & a+3b \\ 8 & 2a+9 \end{pmatrix}$$ ***M1A1***

So require $a = b$ ***M1A1***

[4 marks]

(b.i) Find the value of a if the determinant of matrix M is –1. [2]

Markscheme

$$\begin{vmatrix} 1 & a \\ 2 & 3 \end{vmatrix} = 3 - 2a = -1 \Rightarrow a = 2$$ ***M1A1***

[2 marks]

(b.ii) Write down M^{-1} for this value of a. [1]

Markscheme

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$ ***A1***

[1 mark]

14. [Maximum mark: 9] EXM.1.AHL.TZ0.6

Let $C = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$.

The 2×2 matrix Q is such that $3Q = 2C - D$

(a) Find Q. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$3Q = \begin{pmatrix} -4 & 8 \\ 2 & 14 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$ ***(A1)***

$3Q = \begin{pmatrix} -9 & 6 \\ 3 & 14 - a \end{pmatrix}$ ***(A1)***

$Q = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14-a}{3} \end{pmatrix}$ ***(A1) (N3)***

[3 marks]

(b) Find CD. [4]

Markscheme

$CD = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$

$= \begin{pmatrix} -14 & -4 + 4a \\ -2 & 2 + 7a \end{pmatrix}$ ***(A1)(A1)(A1)(A1) (N4)***

[4 marks]

(c) Find $\boldsymbol{D}^{-1}$. [2]

Markscheme

det $\boldsymbol{D} = 5a + 2$ (may be implied) ***(A1)***

$\boldsymbol{D}^{-1} = \frac{1}{5a+2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$ ***(A1) (N2)***

[2 marks]

15. [Maximum mark: 5] EXM.1.AHL.TZ0.30

Matrices ***A***, ***B*** and ***C*** are defined as

$$\boldsymbol{A} = \begin{pmatrix} 1 & 5 & 1 \\ 3 & -1 & 3 \\ -9 & 3 & 7 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 8 \\ 0 \\ -4 \end{pmatrix}.$$

(a) Given that $\boldsymbol{AB} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$, find a. [1]

Markscheme

$a = 16$ ***A1***

[1 mark]

(b) Hence, or otherwise, find $\boldsymbol{A}^{-1}$. [2]

Markscheme

$\boldsymbol{A}^{-1} = \frac{1}{16}\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$ ***(M1)A1***

[2 marks]

(c) Find the matrix ***X***, such that $\boldsymbol{AX} = \boldsymbol{C}$. [2]

Markscheme

$\boldsymbol{AX} = \boldsymbol{C} \Rightarrow \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{C}$ ***(M1)***

$$= \frac{1}{16}\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 0\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}8\\ 0\\ -4\end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix}12\\ 24\\ -4\end{pmatrix} \quad \left(= \begin{pmatrix}0.75\\ 1.5\\ -0.25\end{pmatrix}\right) \qquad \textit{A1}$$

[2 marks]

16. [Maximum mark: 7] EXM.1.AHL.TZ0.45

(a) Given matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ for which $\boldsymbol{AB} = \boldsymbol{C}$ and $\det \boldsymbol{A} \neq 0$, express $\boldsymbol{B}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{C}$. [2]

Markscheme

Since $\det \boldsymbol{A} \neq 0$, $\boldsymbol{A}^{-1}$ exists. ***(M1)***

Hence $\boldsymbol{AB} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$ ***(C1)***

[2 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$, and $\boldsymbol{C} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$.

(b.i) Find the matrix $\boldsymbol{DA}$. [1]

Markscheme

$$\boldsymbol{DA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ***(A1)***

[1 mark]

(b.ii) Find $\boldsymbol{B}$ if $\boldsymbol{AB} = \boldsymbol{C}$. [2]

Markscheme

$\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{DC}$ ***(M1)***

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$ ***(A1)***

[2 marks]

(c) Find the coordinates of the point of intersection of the planes $x + 2y + 3z = 5, 2x - y + 2z = 7,$ $3x - 3y + 2z = 10.$ [2]

Markscheme

The system of equations is
$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x - y + 2z &= 7 \\ 3x - 3y + 2z &= 10 \end{aligned}$$

or $\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{C}$ ***(M1)***

The required point = (1, −1, 2). ***(A1)***

[2 marks]

17. [Maximum mark: 6] EXM.1.AHL.TZ0.9

The matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$ has inverse $\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & -2 & -2 \\ 3 & 1 & 1 \\ a & 6 & b \end{pmatrix}$.

(a.i) Write down the value of a. [1]

Markscheme

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$a = 4$ ***A1 N1***

[1 mark]

(a.ii) Write down the value of b. [1]

Markscheme

$b = 7$ ***A1 N1***

[1 mark]

Consider the simultaneous equations

$$x + 2y = 7$$

$$-3x + y - z = 10$$

$$2x - 2y + z = -12$$

(b) Write these equations as a matrix equation. [1]

Markscheme

EITHER

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$$ ***A1 N1***

OR

$$\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$$ ***A1 N1***

[1 mark]

(c) Solve the matrix equation. [3]

Markscheme

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$$ (accept algebraic method) ***(M1)***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$ (accept $x = -3$, $y = 5$, $z = 4$) ***A2 N3***

[3 marks]

18. [Maximum mark: 7] EXM.1.AHL.TZ0.26

Consider the matrices

$\boldsymbol{A} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}.$

(a) Find ***BA***. [2]

Markscheme

$\boldsymbol{BA} = \left(\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}\right) = \begin{pmatrix} 18 & -14 \\ -4 & 4 \end{pmatrix}$ ***A2***

Note: Award ***A1*** for one error, ***A0*** for two or more errors.

[2 marks]

(b) Calculate det (***BA***). [2]

Markscheme

det(***BA***) = (72 – 56) = 16 ***(M1)A1***

[2 marks]

(c) Find $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1})\boldsymbol{A}$. [3]

Markscheme

EITHER

$\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1})\boldsymbol{A} = \boldsymbol{BA} + 2\boldsymbol{A}$ ***(M1)A1***

$= \begin{pmatrix} 24 & -18 \\ 6 & -4 \end{pmatrix}$ ***A1***

OR

$A^{-1} = -\frac{1}{2}\begin{pmatrix} -4 & 2 \\ -5 & 3 \end{pmatrix}$ ***(A1)***

an attempt to evaluate ***(M1)***

$A^{-1}B + 2A^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -16 \\ 1 & -21 \end{pmatrix} - \begin{pmatrix} -4 & 2 \\ -5 & 3 \end{pmatrix}$

$A(A^{-1}B + 2A^{-1})A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 4.5 & 7.5 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} 24 & -18 \\ 6 & -4 \end{pmatrix}$ ***A1***

[3 marks]

19. [Maximum mark: 6] EXM.1.AHL.TZ0.4

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$.

(a) Write down the inverse, $\boldsymbol{A}^{-1}$. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

det $\boldsymbol{A} = 5(1) - 7(-2) = 19$

$\boldsymbol{A}^{-1} = \frac{1}{19}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$ ***(A2)***

Note: Award ***(A1)*** for $\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix}$, ***(A1)*** for dividing by 19.

OR

$\boldsymbol{A}^{-1} = \begin{pmatrix} 0.0526 & 0.105 \\ -0.368 & 0.263 \end{pmatrix}$ ***(G2)***

[2 marks]

B, ***C*** and ***X*** are also 2×2 matrices.

(b.i) Given that $\boldsymbol{XA} + \boldsymbol{B} = \boldsymbol{C}$, express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. [2]

Markscheme

$\boldsymbol{XA} + \boldsymbol{B} = \boldsymbol{C} \Rightarrow \boldsymbol{XA} = \boldsymbol{C} - \boldsymbol{B}$ ***(M1)***

$\boldsymbol{X} = (\boldsymbol{C} - \boldsymbol{B})\boldsymbol{A}^{-1}$ ***(A1)***

OR

$X = (C - B)A^{-1}$ ***(A2)***

[2 marks]

(b.ii) Given that $B = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$, find X. [2]

Markscheme

$(C - B)A^{-1} = \begin{pmatrix} -11 & -7 \\ -13 & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$ ***(A1)***

$\Rightarrow X = \begin{pmatrix} \frac{38}{19} & \frac{-57}{19} \\ \frac{-76}{19} & \frac{19}{19} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$ ***(A1)***

OR

$X = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$ ***(G2)***

Note: If premultiplication by $\mathbf{A}^{-1}$ is used, award ***(M1)(M0)*** in part (i) but award ***(A2)*** for $\begin{pmatrix} \frac{-37}{19} & \frac{11}{19} \\ \frac{12}{19} & \frac{94}{19} \end{pmatrix}$ in part (ii).

[2 marks]

20. [Maximum mark: 6] EXM.1.AHL.TZ0.27

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be non-singular 2×2 matrices, $\boldsymbol{I}$ the 2×2 identity matrix and k a scalar. The following statements are **incorrect**. For each statement, write down the correct version of the right hand side.

(a) $(\boldsymbol{A}+\boldsymbol{B})^2=\boldsymbol{A}^2+2\boldsymbol{AB}+\boldsymbol{B}^2$ [2]

Markscheme

$(\boldsymbol{A}+\boldsymbol{B})^2=\boldsymbol{A}^2+\boldsymbol{AB}+\boldsymbol{BA}+\boldsymbol{B}^2$ ***A2***

Note: Award ***A1*** in parts (a) to (c) if error is correctly identified, but not corrected.

[2 marks]

(b) $(\boldsymbol{A}-k\boldsymbol{I})^3=\boldsymbol{A}^3-3k\boldsymbol{A}^2+3k^2\boldsymbol{A}-k^3$ [2]

Markscheme

$(\boldsymbol{A}-k\boldsymbol{I})^3=\mathbf{A}^3-3k\boldsymbol{A}^2+3k^2\boldsymbol{A}-k^3\boldsymbol{I}$ ***A2***

Note: Award ***A1*** in parts (a) to (c) if error is correctly identified, but not corrected.

[2 marks]

(c) $\boldsymbol{CA}=\boldsymbol{B}\ \ \boldsymbol{C}=\frac{B}{A}$ [2]

Markscheme

$\boldsymbol{CA}=\boldsymbol{B}\Rightarrow\boldsymbol{C}=\boldsymbol{BA}^{-1}$ ***A2***

Note: Award ***A1*** in parts (a) to (c) if error is correctly identified, but not corrected.

[2 marks]

21. [Maximum mark: 6] EXM.1.AHL.TZ0.8

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 18 \\ 23 \\ 13 \end{pmatrix}$, and $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Write down the inverse matrix $\boldsymbol{A}^{-1}$. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\boldsymbol{A}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{pmatrix}$ or $\begin{pmatrix} -0.333 & 0.667 & -0.333 \\ -0.333 & 1.67 & -2.33 \\ 0.667 & -1.33 & 1.67 \end{pmatrix}$

A2 N2

[2 marks]

Consider the equation $\boldsymbol{AX} = \boldsymbol{B}$.

(b.i) Express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$. [1]

Markscheme

$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$ ***A1 N1***

[1 mark]

(b.ii) **Hence**, solve for $\boldsymbol{X}$. [3]

Markscheme

$X = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ ***A3 N3***

[3 marks]

22. [Maximum mark: 6] EXM.1.AHL.TZ0.22

Let $A = \begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix}$.

(a) Find ***AB***. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

Attempting to multiply matrices ***(M1)***

$$\begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + x^2 - 2 \\ 9 + x + 8 \end{pmatrix} \left(= \begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} \right)$$

A1A1 N3

[3 marks]

(b) The matrix $C = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$ and 2***AB*** = ***C***. Find the value of x. [3]

Markscheme

Setting up equation ***M1***

eg $2\begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$, $\begin{pmatrix} 2 + 2x^2 \\ 34 + 2x \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$, $\begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$

$$\begin{matrix} 2 + 2x^2 = 20 \\ 34 + 2x = 28 \end{matrix} \quad \begin{pmatrix} 1 + x^2 = 10 \\ 17 + x = 14 \end{pmatrix}$$ ***(A1)***

$x = -3$ ***A1 N2***

[3 marks]

23. [Maximum mark: 6] EXM.1.AHL.TZ0.10

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ k & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find, in terms of k,

(a) $2\boldsymbol{A} - \boldsymbol{B}$. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2\boldsymbol{A} = \begin{pmatrix} 6 & 4 \\ 2k & 8 \end{pmatrix}$ ***(A1)***

$2\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 4 & 2 \\ 2k - 1 & 5 \end{pmatrix}$ ***A2 N3***

[3 marks]

(b) det $(2\boldsymbol{A} - \boldsymbol{B})$. [3]

Markscheme

Evidence of using the definition of determinant ***(M1)***

Correct substitution ***(A1)***

eg $4(5) - 2(2k - 1)$, $20 - 2(2k - 1)$, $20 - 4k + 2$

det $(2\boldsymbol{A} - \boldsymbol{B}) = 22 - 4k$ ***A1 N3***

[3 marks]

24. [Maximum mark: 5] EXM.1.AHL.TZ0.25

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ a & -1 \end{pmatrix}$.

(a) Find the matrix $\boldsymbol{A}^2$. [2]

Markscheme

$\boldsymbol{A}^2 = \begin{pmatrix} 2a & -2 \\ -a & 2a+1 \end{pmatrix}$ ***(M1)A1***

[2 marks]

(b) If $\det \boldsymbol{A}^2 = 16$, determine the possible values of a. [3]

Markscheme

METHOD 1

$\det \boldsymbol{A}^2 = 4a^2 + 2a - 2a = 4a^2$ ***M1***

$a = \pm 2$ ***A1A1 N2***

METHOD 2

$\det \boldsymbol{A} = -2a$ ***M1***

$\det \boldsymbol{A} = \pm 4$

$a = \pm 2$ ***A1A1 N2***

[3 marks]

25. [Maximum mark: 6] EXM.1.AHL.TZ0.29

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a and b are non-zero real numbers.

(a) Show that $\boldsymbol{M}$ is non-singular. [2]

Markscheme

finding det$\boldsymbol{M} = a^2 + b^2$ ***A1***

$a^2 + b^2 > 0$, therefore $\boldsymbol{M}$ is non-singular or equivalent statement ***R1***

[2 marks]

(b) Calculate $\boldsymbol{M}^2$. [2]

Markscheme

$$\boldsymbol{M}^2 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & 2ab \\ -2ab & a^2 - b^2 \end{pmatrix}$$ ***M1A1***

[2 marks]

(c) Show that det($\boldsymbol{M}^2$) is positive. [2]

Markscheme

EITHER

$\det(\boldsymbol{M}^2) = \left(a^2 - b^2\right)\left(a^2 - b^2\right) + (2ab)(2ab)$ ***A1***

$\det(\boldsymbol{M}^2) = \left(a^2 - b^2\right)^2 + (2ab)^2 \quad \left(= \left(a^2 + b^2\right)^2\right)$

since the first term is non-negative and the second is positive ***R1*** therefore det(M2) > 0

Note: Do not penalise first term stated as positive.

OR

$\det(M^2) = (\det M)^2$ ***A1***

since det M is positive so too is det (M^2) ***R1***

[2 marks]

26. [Maximum mark: 6] EXM.1.AHL.TZ0.52

(a) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}$. [2]

Markscheme

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -7 & 3 \\ 0 & 2 & -1 \\ -1 & 3 & -1 \end{pmatrix}$$ ***A2 N2***

[2 marks]

(b) **Hence** solve the system of equations

$$x + 2y + z = 0$$

$$x + y + 2z = 7$$

$$2x + y + z = 17$$ [4]

Markscheme

In matrix form $Ax = B$ or $x = A^{-1}B$ ***M1***

$x = 2, y = -3, z = 4$ ***A1A1A1 N0***

[4 marks]