Matrices - intro [146 marks]

1. [Maximum mark: 4] EXM.1.AHL.TZ0.2

If $A = \begin{pmatrix} 2p & 3 \\ -4p & p \end{pmatrix}$ and det $A = 14$, find the possible values of p. [4]

2. [Maximum mark: 4] EXM.1.AHL.TZ0.3

A and B are 2 × 2 matrices, where $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and

$BA = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find B [4]

3. [Maximum mark: 6] EXM.1.AHL.TZ0.28

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ 2 + \mathrm{e}^x & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of x for which $\boldsymbol{A}$ is singular. [6]

4. [Maximum mark: 5] EXM.1.AHL.TZ0.32

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find k. [5]

5. [Maximum mark: 5] EXM.1.AHL.TZ0.33

Given that $\boldsymbol{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and that $\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = 0$ find k. [5]

6. [Maximum mark: 6] EXM.1.AHL.TZ0.34

The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{X})$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$. [6]

7. [Maximum mark: 3] EXM.1.AHL.TZ0.44

Find the values of the real number k for which the determinant of the matrix $\begin{pmatrix} k-4 & 3 \\ -2 & k+1 \end{pmatrix}$ is equal to zero. [3]

8. [Maximum mark: 3] EXM.1.AHL.TZ0.46

If $\boldsymbol{A} = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}$, find 2 values of x and y, given that $\boldsymbol{AB} = \boldsymbol{BA}$. [3]

9. [Maximum mark: 6] EXM.1.AHL.TZ0.50

Given that $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, find $\boldsymbol{X}$ if $\boldsymbol{BX} = \boldsymbol{A} - \boldsymbol{AB}$.

[6]

10. [Maximum mark: 6] EXM.1.AHL.TZ0.7

Let $\begin{pmatrix} b & 3 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 9 & 5 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ a & 15 \end{pmatrix}$.

(a.i) Write down the value of a. [1]

(a.ii) Find the value of b. [2]

(b) Let $3\begin{pmatrix} -4 & 8 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 \\ q & -4 \end{pmatrix} = \begin{pmatrix} -22 & 24 \\ 9 & 23 \end{pmatrix}$.

Find the value of q. [3]

11. [Maximum mark: 6] EXM.1.AHL.TZ0.35

(a) Write down the inverse of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$$

[2]

(b) **Hence**, find the point of intersection of the three planes.

$$x - 3y + z = 1$$
$$2x + 2y - z = 2$$
$$x - 5y + 3z = 3$$

[3]

(c) A fourth plane with equation $x + y + z = d$ passes through the point of intersection. Find the value of d. [1]

12. [Maximum mark: 4] EXM.1.AHL.TZ0.43

(a) Find the values of a and b given that the matrix

$A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix

$B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$. [2]

(b) For the values of a and b found in part (a), solve the system of linear equations

$$\begin{aligned} x + 2y - 2z &= 5 \\ 3x + by + z &= 0 \\ -x + y - 3z &= a - 1. \end{aligned}$$

[2]

13. [Maximum mark: 7] EXM.1.AHL.TZ0.16

(a) Find a relationship between a and b if the matrices $M = \begin{pmatrix} 1 & a \\ 2 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & b \\ 2 & 3 \end{pmatrix}$ commute under matrix multiplication. [4]

(b.i) Find the value of a if the determinant of matrix M is –1.

[2]

(b.ii) Write down M^{-1} for this value of a. [1]

14. [Maximum mark: 9] EXM.1.AHL.TZ0.6

Let $\boldsymbol{C} = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$.

The 2×2 matrix $\boldsymbol{Q}$ is such that $3\boldsymbol{Q} = 2\boldsymbol{C} - \boldsymbol{D}$

(a) Find $\boldsymbol{Q}$. [3]

(b) Find $\boldsymbol{CD}$. [4]

(c) Find $\boldsymbol{D}^{-1}$. [2]

15. [Maximum mark: 5] EXM.1.AHL.TZ0.30

Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined as

$$\boldsymbol{A} = \begin{pmatrix} 1 & 5 & 1 \\ 3 & -1 & 3 \\ -9 & 3 & 7 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 8 \\ 0 \\ -4 \end{pmatrix}.$$

(a) Given that $\boldsymbol{AB} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$, find a. [1]

(b) Hence, or otherwise, find $\boldsymbol{A}^{-1}$. [2]

(c) Find the matrix $\boldsymbol{X}$, such that $\boldsymbol{AX} = \boldsymbol{C}$. [2]

16. [Maximum mark: 7] EXM.1.AHL.TZ0.45

(a) Given matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ for which $\boldsymbol{AB} = \boldsymbol{C}$ and $\det \boldsymbol{A} \neq 0$, express $\boldsymbol{B}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{C}$. [2]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$, and $\boldsymbol{C} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$.

(b.i) Find the matrix $\boldsymbol{DA}$. [1]

(b.ii) Find $\boldsymbol{B}$ if $\boldsymbol{AB} = \boldsymbol{C}$. [2]

(c) Find the coordinates of the point of intersection of the planes $x + 2y + 3z = 5, 2x - y + 2z = 7,$ $3x - 3y + 2z = 10.$ [2]

17. [Maximum mark: 6] EXM.1.AHL.TZ0.9

The matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$ has inverse $\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & -2 & -2 \\ 3 & 1 & 1 \\ a & 6 & b \end{pmatrix}$.

(a.i) Write down the value of a. [1]

(a.ii) Write down the value of b. [1]

Consider the simultaneous equations

$$x + 2y = 7$$

$$-3x + y - z = 10$$

$$2x - 2y + z = -12$$

(b) Write these equations as a matrix equation. [1]

(c) Solve the matrix equation. [3]

18. [Maximum mark: 7] EXM.1.AHL.TZ0.26

Consider the matrices

$\boldsymbol{A} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}$.

(a) Find $\boldsymbol{BA}$. [2]

(b) Calculate det ($\boldsymbol{BA}$). [2]

(c) Find $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1})\boldsymbol{A}$. [3]

19. [Maximum mark: 6] EXM.1.AHL.TZ0.4

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$.

(a) Write down the inverse, $\boldsymbol{A}^{-1}$. [2]

$\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{X}$ are also 2×2 matrices.

(b.i) Given that $\boldsymbol{XA} + \boldsymbol{B} = \boldsymbol{C}$, express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. [2]

(b.ii) Given that $\boldsymbol{B} = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$, and $\boldsymbol{C} = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$, find $\boldsymbol{X}$. [2]

20. [Maximum mark: 6] EXM.1.AHL.TZ0.27

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be non-singular 2×2 matrices, $\boldsymbol{I}$ the 2×2 identity matrix and k a scalar. The following statements are **incorrect**. For each statement, write down the correct version of the right hand side.

(a) $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$ [2]

(b) $(\boldsymbol{A} - k\boldsymbol{I})^3 = \boldsymbol{A}^3 - 3k\boldsymbol{A}^2 + 3k^2\boldsymbol{A} - k^3$ [2]

(c) $\boldsymbol{CA} = \boldsymbol{B}$ $\boldsymbol{C} = \frac{B}{A}$ [2]

21. [Maximum mark: 6] EXM.1.AHL.TZ0.8

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 18 \\ 23 \\ 13 \end{pmatrix}$, and $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Write down the inverse matrix $\boldsymbol{A}^{-1}$. [2]

Consider the equation $\boldsymbol{AX} = \boldsymbol{B}$.

(b.i) Express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$. [1]

(b.ii) **Hence**, solve for $\boldsymbol{X}$. [3]

22. [Maximum mark: 6] EXM.1.AHL.TZ0.22

Let $\boldsymbol{A} = \begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix}$.

(a) Find $\boldsymbol{AB}$. [3]

(b) The matrix $\boldsymbol{C} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$ and $2\boldsymbol{AB} = \boldsymbol{C}$. Find the value of x. [3]

23. [Maximum mark: 6] EXM.1.AHL.TZ0.10

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ k & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find, in terms of k,

(a) $2\boldsymbol{A} - \boldsymbol{B}$. [3]

(b) $\det(2\boldsymbol{A} - \boldsymbol{B})$. [3]

24. [Maximum mark: 5] EXM.1.AHL.TZ0.25

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ a & -1 \end{pmatrix}$.

(a) Find the matrix $\boldsymbol{A}^2$. [2]

(b) If $\det \boldsymbol{A}^2 = 16$, determine the possible values of a. [3]

25. [Maximum mark: 6] EXM.1.AHL.TZ0.29

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a and b are non-zero real numbers.

(a) Show that $\boldsymbol{M}$ is non-singular. [2]

(b) Calculate $\boldsymbol{M}^2$. [2]

(c) Show that $\det(\boldsymbol{M}^2)$ is positive. [2]

26. [Maximum mark: 6] EXM.1.AHL.TZ0.52

(a) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}$. [2]

(b) **Hence** solve the system of equations

$$x + 2y + z = 0$$

$$x + y + 2z = 7$$

$$2x + y + z = 17$$

[4]