Name:
Result:

1. *(6 points)*

Consider complex numbers $z = 3 + i$ and $w = 2\text{cis}\frac{\pi}{5}$.

(a) Write z in polar form and w in Cartesian form. [2]

Consider another complex number $c = 1 - 2i$.

(b) Calculate $z \cdot c$, express your answer in Cartesian form. [1]

(c) Describe the transformation that takes place when c is multiplied by z. [2]

(d) Describe the transformation that takes place when when w is added to c. [1]

(a) $z = \sqrt{10}\text{cis}(0.322)$, $w = 1.62 + 1.18i$.

(b) $(3 + i)(1 - 2i) = 5 - 5i$.

(c) Stretch by a factor of $\sqrt{10}$, counter-clockwise rotation by 0.322.

(d) Translation by $\binom{1.62}{1.18}$.

2. *(6 points)*

Three AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as:

$$V_1 = 20\sin(2t + 20^\circ), \qquad V_2 = 30\sin(2t + 40^\circ), \qquad V_3 = 40\sin(2t + 60^\circ)$$

The combined voltage can be expressed in the form:

$$V_{\text{total}} = A\sin(bt + \theta^\circ)$$

(a) State the value of b [1]

(b) Determine the values of A and θ. [4]

(c) State the maximum combined voltage. [1]

(a) $b = 2$.

(b)

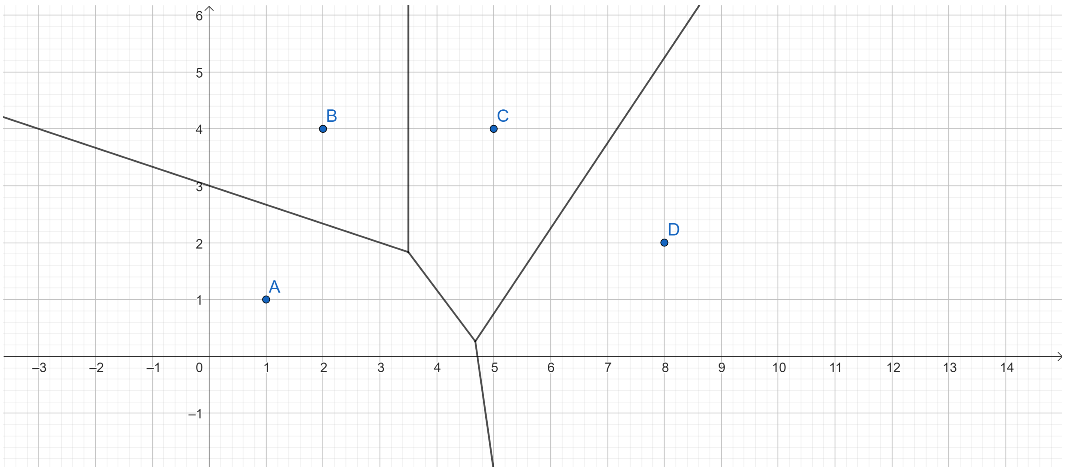
$$\begin{aligned} & 20\sin(2t + 20^\circ) + 30\sin(2t + 40^\circ) + 40\sin(2t + 60^\circ) = \\ &= \operatorname{Im}(20e^{(2t+\frac{\pi}{9})i} + 30e^{(2t+\frac{2\pi}{9})i} + 40e^{(2t+\frac{\pi}{3})i}) = \\ &= \operatorname{Im}(10e^{2ti}(2e^{i\frac{\pi}{9}} + 3e^{i\frac{2\pi}{9}} + 4e^{i\frac{\pi}{3}})) = \\ &= \operatorname{Im}(10e^{2ti}(8.665197e^{0.777155i})) = \\ &= \operatorname{Im}(86.7e^{(2t+0.777155)i}) = 86.7\sin(2t + 0.777155) = 86.7\sin(2t + 44.5^\circ) \end{aligned}$$

So $A = 86.7$ and $\theta = 44.5^\circ$.

(c) 86.7

3. *(6 points)*

Consider the following Voronoi diagram for sites A, B, C and D.



(a) State the equation of the edge between B and C. $x = 3.5$ [1]

(b) Find the equation of the edge between C and D, give your answer in the form $Ax+By+C=0$, where $A, B, C \in \mathbb{Z}$. [2]

$\overrightarrow{CD} = \binom{3}{-2}$ and $M_{CD} = (6.5, 3)$, so the equation is:

$$3x - 2y - 13.5 = 0$$

All coefficients must be integers, so we get:

$$6x - 4y - 27 = 0$$

(c) A new site E is to be added at $(8, 4)$. Complete the Voronoi diagram with the new site added. [2]

The new Voronoi diagram should include the new site, the bisectors between C and E and between D and E and part of the bisector between CD.

(d) State the coordinates of the point, which is equidistant to sites C, D and E. [1]

$(6.5, 3)$

4. *(7 points)*

Consider points $A(1,0,1), B(0,3,1)$ and $C(2,5,3)$.

(a) Calculate $\overrightarrow{AB} \times \overrightarrow{AC}$ and hence find the area of the triangle ABC. [3]

(b) Point D is on the line segment AB such that AB is perpendicular to CD. Use your answer to part (a) to find the length of CD. [2]

(c) Find the angle that the triangle makes with the plane $z = 1$. [2]

(a) $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix}$.

Area $= \frac{1}{2} \left| \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix} \right| = \frac{1}{2}\sqrt{104} \approx 5.10$.

(b) $|AB| = \sqrt{10}$. So using the area:

$$\frac{1}{2}\sqrt{10}|CD| = \frac{1}{2}\sqrt{104}$$

Which gives $|CD| \approx 3.22$

(c) Using the triangle CDP where P is the perpendicular projection of C onto the $z = 1$ plane we have:

$$\sin\theta = \frac{2}{|CD|}$$

This gives $\theta \approx 0.669 = 38.3°$.

5. *(17 points)*

A box contains 10 balls: 2 red, 3 blue and 5 green. A game consists of picking 3 balls (without replacement), for each blue ball picked the player wins 10 tokens. If however the player picks a red ball, all his winnings are lost.

(a) Find the probability that the player picks 3 blue balls. Express your answer as a fraction in simplest terms. $\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{120}$ [2]

(b) Show that the probability that the player picks 2 blue and 1 green ball is. $\frac{1}{8}$. [2]

$3 \cdot \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{5}{8} = \frac{1}{8}$

The probability that the player picks 1 blue and 2 green balls is $\frac{1}{4}$. Let X be the winnings of the player.

(c) Complete the distribution table for X: [3]

x	0	10	20	30
$P(X=x)$	$\frac{37}{60}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{120}$

(d) Calculate $E(X)$ $E(X) = \frac{37}{60} \cdot 0 + \frac{1}{4} \cdot 10 + \frac{1}{8} \cdot 20 + \frac{1}{120} \cdot 30 = 5.25$ [2]

(e) Hence state how much the game should cost for it to be fair. 5.25 [1]

(f) If Tomasz plays the game 10 times, find the probability that: [4]

Let Y be the number of times Tomasz wins tokens. Then $Y \sim B(10, \frac{23}{60})$

(i) he wins tokens on more than half of these games. $P(Y > 5) = 1 - P(Y \leq 5) = 0.140$

(ii) he wins tokens exactly 7 times, given that he won tokens on more than half of his games. $P(Y = 7 | Y > 5) = \frac{P(Y=7 \cap Y>5)}{P(Y>5)} = \frac{P(Y=7)}{P(Y>5)} = \frac{0.03422699...}{0.1397327...} \approx 0.245$

(g) Tomasz gets bored, if he doesn't win. He will continue playing until a third game in which he doesn't win occurs. Find the probability that he will play exactly 10 games. [3]

He has to lose two games in the first nine and then lose the tenth game. Let Z be the number of loses in the first nine games, then $Z \sim B(9, \frac{37}{60})$. We want:

$$P(Z = 2) \cdot \frac{37}{60} \approx 0.0103$$

6. *(10 points)*

In this question, i denotes a unit vector due east, j denotes a unit vector due north.
The positions of two yachts *Amelia* and *Borubar* are given by the equations:

$$r_A = (2+t)i + (3+2t)j \qquad r_B = (-1+4t)i + 3tj$$

t is measured in hours since 7 am and distances are measured in kilometres.

It makes sense to rewrite the positions as follows:

$$r_A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad r_B = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(a) Find the speed of *Borubar*. $\left|\binom{4}{3}\right| = 5\frac{km}{h}$ [1]

(b) Find the distance between the two yachts at 9 am. [2]

The two yachts are at $A(4,7)$ and $B(7,6)$ respectively. The distance between the two points is $\sqrt{10}$.

(c) Find the shortest distance between the two yachts and the time at which it occurs. Give your answer to the nearest minute. [3]

At any time t the yachts are at $A(2+t, 3+2t)$ and $B(-1+4t, 3t)$ respectively. The distance between these two points is:

$$d(t) = \sqrt{(3t-3)^2 + (t-3)^2}$$

Using the GDC we get that $d_{min} \approx 1.90$ and occurs at $t = 1.2$, so at 8:12.

At 10 am a speedboat sets off from *Amelia* and travels in a straight line. It will be reach *Borubar* at exactly 10:06 am.

At 10 the *Amelia* is at $(5,9)$ and at 10:06 the *Borubar* will be at $(11.4, 9.3)$. So the speedboat needs to travel in the direction of $\binom{6.4}{0.3}$

(e) Find the velocity vector of the speedboat. [2]

The speedboat needs to move by the vector $\binom{6.4}{0.3}$ in 6 minutes, so its velocity vector is $10 \cdot \binom{6.4}{0.3} = \binom{64}{3}$ (because times is measured in hours).

(f) Find the bearing at which it travels. [2]

Using appropriate right triangle with sides (3 and 64) and remembering that bearing is measured clockwise from North, we get $\tan\theta = \frac{64}{3}$, so $\theta \approx 1.52 = 87.3°$.

7. *(16 points)*

The number of cars passing through a certain toll gate on a motorway follows Poisson distribution with 7 cars passing every 5 minutes.

(a) Find the probability that more than 48 cars will pass through the gate between 8:00 and 8:30. [2]

N - number of cars passing in 30 minutes, then $N \sim Po(42)$ and $P(N > 48) = 1 - P(N \leq 48) = 0.158$.

(b) Find the probability that there will be more than 8 cars passing in at least four of the six 5-minute intervals between 8:00 and 8:30 (8:00 - 8:05, 8:05 - 8:10 etc.) [3]

X - number of cars passing in 5 minutes, then $X \sim Po(7)$ and

$$P(X > 8) = 1 - P(X \leq 8) = 0.2709087...$$

I - number of 5-minute intervals, in which more than 8 cars passed, then $I \sim B(6, 0.2709087...)$

$$P(I \geq 4) = 1 - P(I \leq 3) \approx 0.0497$$

Each car passing through the gate pays a toll of 20 PLN. The toll machine can process at most 4 payments per minute. Let Y be the total toll paid by all passing cars during 1 minute.

(c) Explain why Y does not follow Poisson distribution. [1]

Y has an upper limit **or** Y cannot take all natural values (for example Y cannot be 1) **or** a constant multiple of Poisson variable is not a Poisson variable.

(d) Calculate $P(Y = 20)$. [3]

W - number of cars passing in 1 minute, then $W \sim Po(1.4)$. $P(Y = 20) = P(W = 1) = 0.345$.

(e) Show that $E(Y) > 21$ [4]

x	0	20	40	60	80
$P(X = x)$	...	0.345	0.242	0.113	...

$E(Y) > 0.345 \cdot 20 + 0.242 \cdot 40 + 0.113 \cdot 60 = 23.36 > 21$

The number of cars passing through a different toll gate on another motorway also follows Poisson distribution. On average 9 cars pass every 10 minutes. Let Z be the total number of cars passing through the two gates in 20 minutes.

(f) Calculate the probability that Z is at most 40. State any assumptions that you've made in your calculations. [3]

The assumption is that the numbers of cars passing through the two gates are independent. Then the sum of Poisson variables is also a Poisson variable, so $Z \sim Po(46)$.

$$P(Z \leq 40) = 0.211$$