Name:

Mathematics preIB Examination

January 9, 2023

1 hour 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Calculators are **not allowed** for this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is [**74 marks**].
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.
- Write your solutions in the space provided.

1. [Maximum mark: 11]

Consider the function $f(x) = 3 - 2\sin\left(\frac{\pi}{6}x\right)$, where $0 \leqslant x \leqslant 24$.

a) Find the period of the function. [2]

b) Find the exact value of $f(2)$. [2]

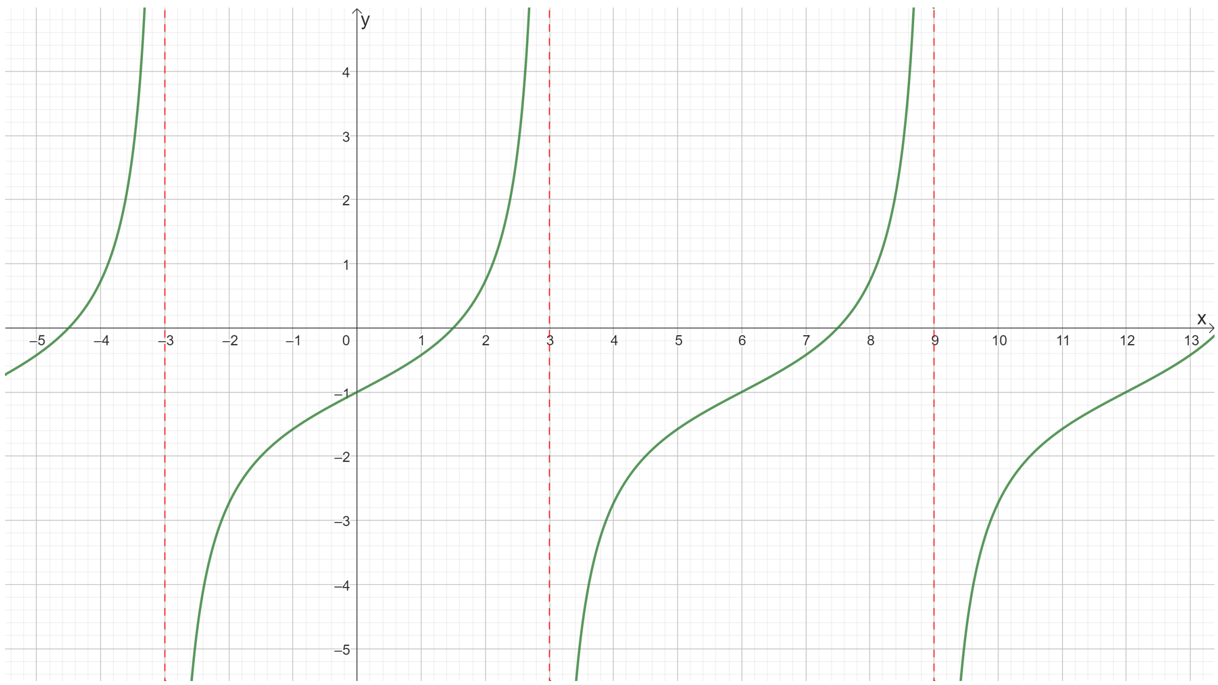
c) Find the range of values of the function. [1]

d) Sketch the graph of $y = f(x)$. [2]

e) Solve the inequality $f(x) > 2$. [4]

2. [Maximum mark: 15]

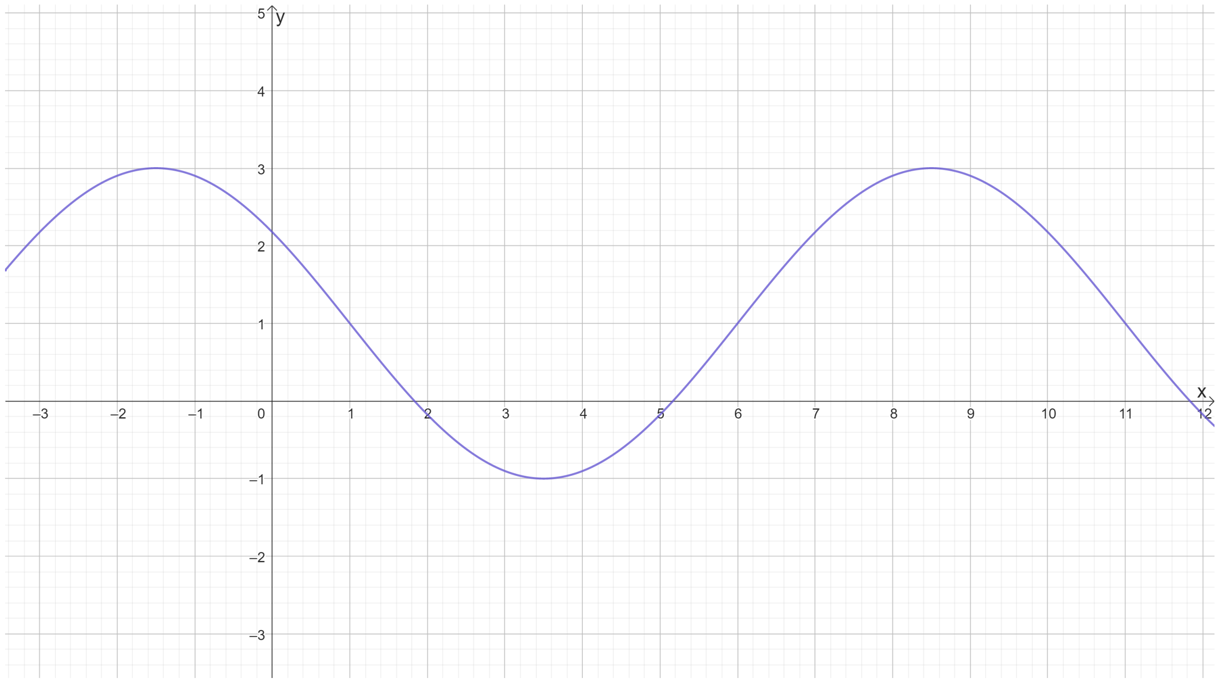
a) Part of the graph of $f(x) = \tan(Bx) + D$ is shown below. The graph crosses the y-axis at $(0, -1)$ and has vertical asymptotes at $x = 3 + 6k$, where $k \in \mathbb{Z}$ [4]



(i) Find the constants B and D.

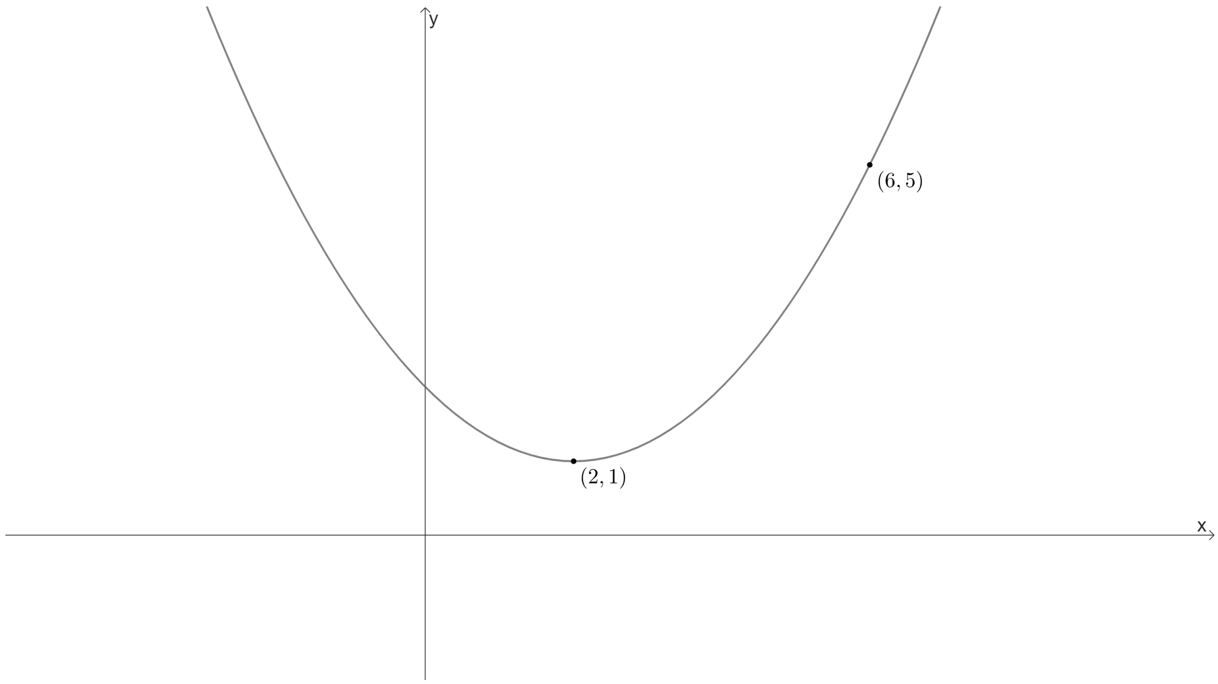
(ii) Write down the transformations that map the graph of $y = \tan(x)$ onto the graph of $y = f(x)$.

b) Part of the graph of $f(x) = A\sin(B(x - C)) + D$ is shown below. The graph has a minimum at $(3.5, -1)$ and a maximum at $(8.5, 3)$. [4]



Find the constants A, B, C and D, given that $0 < C < 5$.

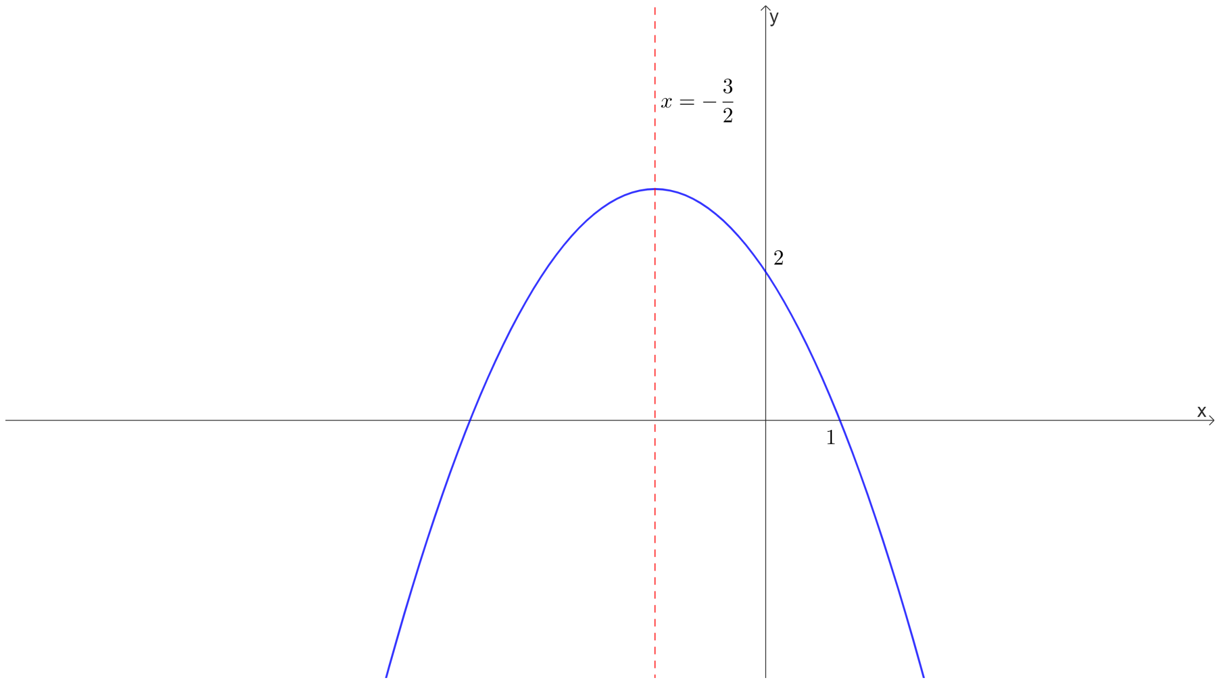
c) The diagram below shows the graph of a quadratic function. The graph has a minimum at $(2, 1)$ and passes through $(6, 5)$. [4]



(i) Find the equation of the function in the form $f(x) = ax^2 + bx + c$.

(ii) Write down a sequence of transformations that maps the graph of $y = x^2$ onto the graph of $y = f(x)$.

d) The diagram below shows the graph of a quadratic function. The line $x = -\frac{3}{2}$ is the axis of symmetry of the graph. The graph has a y-intercept at $y = 2$ and one of its x-intercepts at $x = 1$. [3]



Find the equation of the function in the form $y = ax^2 + bx + c$.

3. [Maximum mark: 12]

Consider the equation:

$$x^2 + (m-1)x + m + 7 = 0$$

Let the solutions to the above equations be α and β.

a) First consider the case where $m = -4$.

i) Find the value of $\alpha^2 + \beta^2$. [3]

ii) Find a quadratic equation in the form $x^2 + Bx + C = 0$, whose solutions are α^2 and β^2, where B and C are integers to be found. [2]

b) Now let $m \in \mathbb{R}$.

i) Find $(\alpha - 1)(\beta - 1)$ in terms of m. [2]

ii) Hence, or otherwise, find the set of values of m, for which the equations has two distinct real solutions which are both greater than 1. [5]

4. [Maximum mark: 13]

Solve the following equations:

a) $\left(\dfrac{1}{x+1}\right)^2 + \dfrac{1}{x+1} - 6 = 0,$ [4]

b) $\cos^2(3\theta) + 2\cos(3\theta) = 3$, where $0 \leqslant \theta \leqslant 2\pi$, [4]

c) $3\sin\theta + 3 = 2\cos^2\theta$, where $0 \leqslant \theta \leqslant 2\pi$. [5]

5. [Maximum mark: 12]

Let θ be such that $\pi < \theta < 2\pi$ and $\tan\theta = 4$.

a) Find the exact values of $\sin\theta$ and $\cos\theta$. [4]

b) Calculate the exact value of $\dfrac{\sin(\pi-\theta)+\tan(2\pi+\theta)}{\cos(-\theta)\times\sin(2\pi-\theta)}$ [4]

c) Prove the identity [4]

$$\frac{1}{1+\cos\alpha}+\frac{1}{1-\cos\alpha}\equiv\frac{2}{\sin^2\alpha}$$

and hence find the value of $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\cos\theta}$.

6. [Maximum mark: 11]

Let $f(x) = \frac{1}{2}x^2 + x - \frac{3}{2}$, where $x \in \mathbb{R}$.

a) Sketch the graph of $y = f(x)$. Clearly indicate axes intercepts and the coordinates of the vertex. [3]

b) State the set of all possible value of p, for which the equation

$$f(x) = p$$

has two distinct real solutions. [1]

c) On the same set of axes sketch the graph of $y = -\frac{1}{2}x + \frac{7}{2}$. Clearly indicate axes intercepts and the coordinates of points of intersections of the two graphs. [3]

d) Find the set of all possible value of c, for which the line $y = -\frac{1}{2}x + c$ intersects the parabola $y = f(x)$ twice. [4]