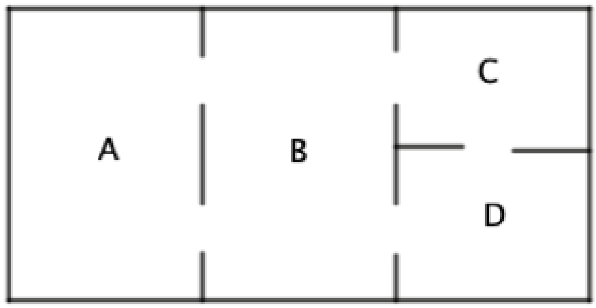
Markov chains [69 marks]

1. [Maximum mark: 5] EXN.1.AHL.TZ0.5

A robot moves around the maze shown below.



Whenever it leaves a room it is equally likely to take any of the exits.

The time interval between the robot entering and leaving a room is the same for all transitions.

(a) Find the transition matrix for the maze. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 0.25 & 0 & 0.5 \\ 0 & 0.25 & 0.5 & 0 \end{pmatrix} \end{array}$$

(M1)A1A1

Note: Award **A1A0** if there is one error in the matrix. **A0A0** for more than one error.

[3 marks]

(b) A scientist sets up the robot and then leaves it moving around the maze for a long period of time.

Find the probability that the robot is in room B when the scientist returns. [2]

Markscheme

Steady state column matrix is $\begin{pmatrix} 0.2 \\ 0.4 \\ 0.2 \\ 0.2 \end{pmatrix}$ **(M1)**

Probability it is in room B is 0.4 **A1**

[2 marks]

2. [Maximum mark: 7] EXM.1.AHL.TZ0.17

Sue sometimes goes out for lunch. If she goes out for lunch on a particular day then the probability that she will go out for lunch on the following day is 0.4. If she does not go out for lunch on a particular day then the probability she will go out for lunch on the following day is 0.3.

(a) Write down the transition matrix for this Markov chain. [2]

Markscheme

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix}$$ ***M1A1***

[2 marks]

(b) We know that she went out for lunch on a particular Sunday, find the probability that she went out for lunch on the following Tuesday. [2]

Markscheme

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.34 \\ 0.66 \end{pmatrix}$$ ***M1***

So probability is 0.34 ***A1***

[2 marks]

(c) Find the steady state probability vector for this Markov chain. [3]

Markscheme

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix} \Rightarrow 0.4p + 0.3\left(1-p\right) = p \Rightarrow p = \frac{1}{3}$$

M1A1

So vector is $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ ***A1***

[or by investigating high powers of the transition matrix]

[3 marks]

3. [Maximum mark: 5] 22M.1.AHL.TZ1.11

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a,\ b \in \mathbb{Z}$. [3]

Markscheme

$\lambda = 1$

$\begin{pmatrix} -0.8 & 0.7 \\ 0.8 & -0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

(M1)

$0.8x = 0.7y$ ***(A1)***

an eigenvector is $\begin{pmatrix} 7 \\ 8 \end{pmatrix}$ (or equivalent with integer values) ***A1***

[3 marks]

A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^{+}$. [2]

Markscheme

EITHER

(the long-term probability matrix is given by the eigenvector corresponding to the eigenvalue equal to 1, scaled so that the sum of the entries is 1)

$8+7=15$ ***(M1)***

OR

$$\begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix}=\begin{pmatrix} p \\ 1-p \end{pmatrix}$$ ***(M1)***

OR

considering high powers of the matrix *e.g.* $\begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}^{50}$ ***(M1)***

$$\begin{pmatrix} \frac{7}{15} & \frac{7}{15} \\ \frac{8}{15} & \frac{8}{15} \end{pmatrix}$$

THEN

probability of being in state A is $\frac{7}{15}$ ***A1***

[2 marks]

4. [Maximum mark: 12] 22M.2.AHL.TZ2.5

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the $n\text{th}$ time, and Z_n is the probability of it being in another state after dividing for the $n\text{th}$ time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a.i) Write down the value of b. [1]

Markscheme

0.02 ***A1***

[1 mark]

(a.ii) What does b represent in this context? [1]

Markscheme

the probability of mutating from 'not normal state' to 'normal state' ***A1***

Note: The ***A1*** can only be awarded if it is clear that transformation is **from** the mutated state.

[1 mark]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

Markscheme

$\det\begin{pmatrix} 0.94-\lambda & 0.02 \\ 0.06 & 0.98-\lambda \end{pmatrix} = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$ has been used is sufficient for the ***(M1)***.

$(0.94-\lambda)(0.98-\lambda) - 0.0012 = 0$ **OR** $\lambda^2 - 1.92\lambda + 0.92 = 0$ ***(A1)***

$\lambda = 1,\ 0.92\ \left(\frac{23}{25}\right)$ ***A1***

[3 marks]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

Markscheme

$\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.92\begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

Note: Award ***M1*** can be awarded for attempting to find either eigenvector.

$0.02y - 0.06x = 0$ **OR** $0.02y + 0.02x = 0$

$\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ ***A1A1***

Note: Accept any multiple of the given eigenvectors.

[3 marks]

The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(d.i) when $n = 5$. [2]

Markscheme

$\begin{pmatrix}0.94 & 0.02\\0.06 & 0.98\end{pmatrix}^5\begin{pmatrix}1\\0\end{pmatrix}$ **OR** $\begin{pmatrix}0.744 & 0.0852\\0.256 & 0.915\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ ***(M1)***

Note: Condone omission of the initial state vector for the ***M1***.

$0.744 \quad (0.744311\ldots)$ ***A1***

[2 marks]

(d.ii) in the long term. [2]

Markscheme

$\begin{pmatrix}0.25\\0.75\end{pmatrix}$ ***(A1)***

Note: Award ***A1*** for $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{pmatrix}$ seen.

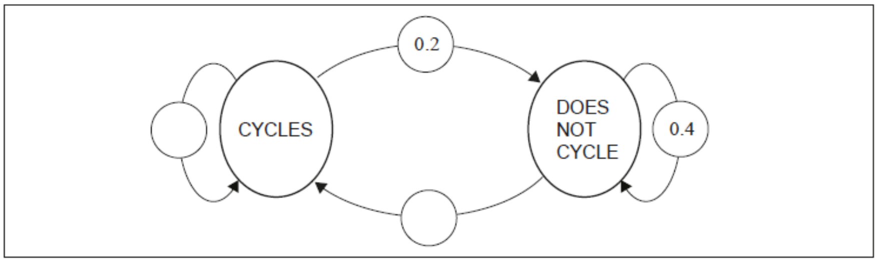
0.25 ***A1***

[2 marks]

5. [Maximum mark: 5] 21N.1.AHL.TZ0.9

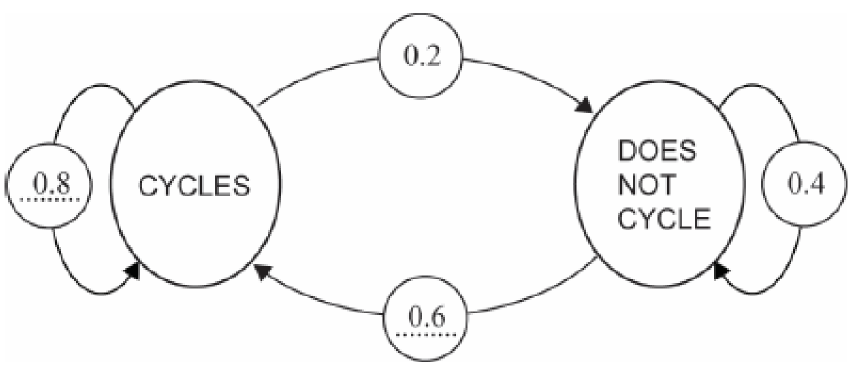
Katie likes to cycle to work as much as possible. If Katie cycles to work one day then she has a probability of 0.2 of not cycling to work on the next work day. If she does not cycle to work one day then she has a probability of 0.4 of not cycling to work on the next work day.

(a) Complete the following transition diagram to represent this information.



[2]

Markscheme



A1A1

[2 marks]

(b) Katie works for 180 days in a year.

Find the probability that Katie cycles to work on her final working day of the year. [3]

Markscheme

$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$$ *(A1)*

$$\boldsymbol{A}^{180} = \begin{pmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{pmatrix}$$ *(M1)*

0.75 *A1*

[3 marks]

6. [Maximum mark: 14] SPM.2.AHL.TZ0.6

A city has two cable companies, X and Y. Each year 20 % of the customers using company X move to company Y and 10 % of the customers using company Y move to company X. All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movements between the two companies in a particular year. [2]

Markscheme

$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$ ***M1A1***

[2 marks]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$. [4]

Markscheme

$\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$ ***M1***

$\lambda = 1$ and 0.7 ***A1***

eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(M1)A1***

Note: Accept any scalar multiple of the eigenvectors.

[4 marks]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [2]

Markscheme

EITHER

$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$ ***A1A1***

OR

$P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ $D = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1A1***

[2 marks]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

Markscheme

$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ ***A1***

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1200 \\ 1200 \end{pmatrix}$ ***M1A1***

attempt to multiply matrices ***M1***

so in company A, after n years, $400\left(2 + 0.7^n\right)$ ***A1***

[5 marks]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

Markscheme

$400 \times 2 = 800$ ***A1***

[1 mark]

7. [Maximum mark: 8] 21M.1.AHL.TZ2.13

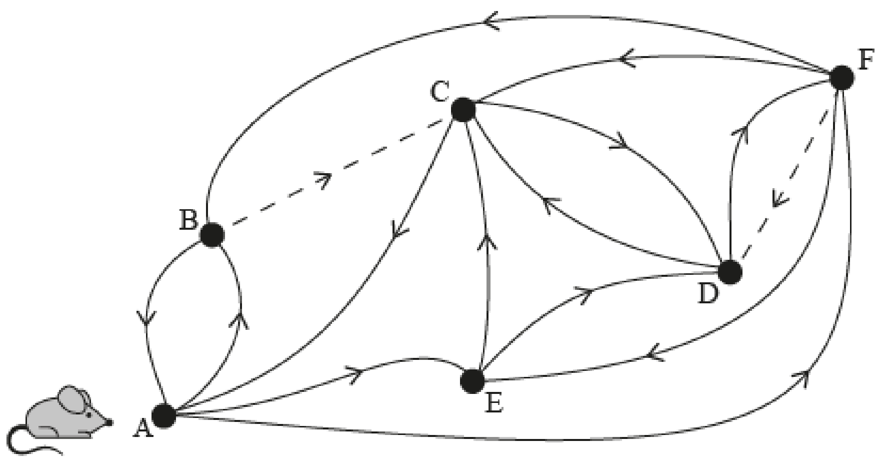
The graph below shows a small maze, in the form of a network of directed routes. The vertices A to F show junctions in the maze and the edges show the possible paths available from one vertex to another.

A mouse is placed at vertex A and left to wander the maze freely. The routes shown by dashed lines indicate paths sprinkled with sugar.

When the mouse reaches any junction, she rests for a constant time before continuing.

At any junction, it may also be assumed that

- the mouse chooses any available normal path with equal probability
- if the junction includes a path sprinkled with sugar, the probability of choosing this path is twice that of a normal path.



(a) Determine the transition matrix for this graph. [3]

Markscheme

transition matrix is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & \frac{2}{3} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \end{array}$$

M1A1A1

Note: Allow the transposed matrix.

Award ***M1*** for a 6×6 matrix with all values between 0 and 1, and all columns (or rows if transposed) adding up to 1, award ***A1*** for one correct row (or column if transposed) and ***A1*** for all rows (or columns if transposed) correct.

[3 marks]

(b) If the mouse was left to wander indefinitely, use your graphic display calculator to estimate the percentage of time that the mouse would spend at point F. [3]

Markscheme

attempting to raise the transition matrix to a large power ***(M1)***

steady state vector is $\begin{pmatrix} (0.157) \\ (0.0868) \\ (0.256) \\ (0.241) \\ (0.0868) \\ 0.173 \end{pmatrix}$ ***(A1)***

so percentage of time spent at vertex F is 17.3% ***A1***

Note: Accept 17.2%.

[3 marks]

(c) Comment on your answer to part (b), referring to at least one limitation of the model. [2]

Markscheme

the model assumes instantaneous travel from junction to junction, ***R1***
and hence the answer obtained would be an overestimate ***R1***

OR

the mouse may eat the sugar over time ***R1***
and hence the probabilities would change ***R1***

Note: Accept any other sensible answer.

[3 marks]

8. [Maximum mark: 13] 21M.2.AHL.TZ1.5

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}.$$

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

Markscheme

finding $\boldsymbol{T}^3$ **OR** use of tree diagram *(M1)*

$$\boldsymbol{T}^3 = \begin{pmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{pmatrix}$$

the probability of sunny in three days' time is 0.65 ***A1***

[2 marks]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

Markscheme

attempt to find eigenvalues *(M1)*

Note: Any indication that $\det(\boldsymbol{T} - \lambda \boldsymbol{I}) = 0$ has been used is sufficient for the *(M1)*.

$$\begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$$

$(\lambda^2 - 1.5\lambda + 0.5 = 0)$

$\lambda = 1,\ \lambda = 0.5$ ***A1***

attempt to find either eigenvector ***(M1)***

$0.8x + 0.3y = x \Rightarrow -0.2x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

A1

$0.8x + 0.3y = 0.5x \Rightarrow 0.3x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

A1

Note: Accept multiples of the stated eigenvectors.

[5 marks]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c.i) Write down the matrix $\boldsymbol{P}$. [1]

Markscheme

$\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ **OR** $\boldsymbol{P} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ ***A1***

Note: Examiners should be aware that different, correct, matrices $\boldsymbol{P}$ may be seen.

[1 mark]

(c.ii) Write down the matrix $\boldsymbol{D}$. [1]

Markscheme

$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$ **OR** $\boldsymbol{D} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

Note: $\boldsymbol{P}$ and $\boldsymbol{D}$ must be consistent with each other.

[1 mark]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

Markscheme

$0.5^n \to 0$ ***(M1)***

$\boldsymbol{D}^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **OR** $\boldsymbol{D}^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

Note: Award ***A1*** only if their $\boldsymbol{D}^n$ corresponds to their $\boldsymbol{P}$

$\boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$ ***(M1)***

60% ***A1***

[4 marks]