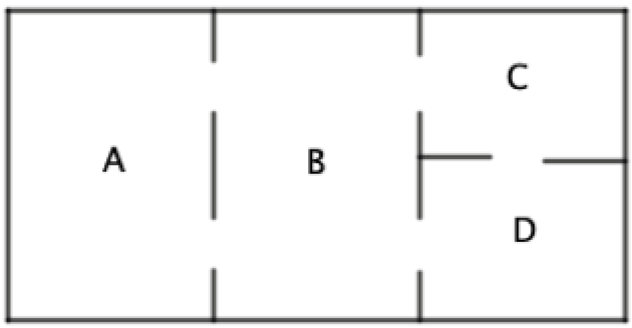
Markov chains [69 marks]

1. [Maximum mark: 5] EXN.1.AHL.TZ0.5

A robot moves around the maze shown below.



Whenever it leaves a room it is equally likely to take any of the exits.

The time interval between the robot entering and leaving a room is the same for all transitions.

(a) Find the transition matrix for the maze. [3]

(b) A scientist sets up the robot and then leaves it moving around the maze for a long period of time.

Find the probability that the robot is in room B when the scientist returns. [2]

2. [Maximum mark: 7] EXM.1.AHL.TZ0.17

Sue sometimes goes out for lunch. If she goes out for lunch on a particular day then the probability that she will go out for lunch on the following day is 0.4. If she does not go out for lunch on a particular day then the probability she will go out for lunch on the following day is 0.3.

(a) Write down the transition matrix for this Markov chain. [2]

(b) We know that she went out for lunch on a particular Sunday, find the probability that she went out for lunch on the following Tuesday. [2]

(c) Find the steady state probability vector for this Markov chain. [3]

3. [Maximum mark: 5] 22M.1.AHL.TZ1.11

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a,\ b \in \mathbb{Z}$. [3]

A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^{+}$. [2]

4. [Maximum mark: 12] 22M.2.AHL.TZ2.5

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the $n\text{th}$ time, and Z_n is the probability of it being in another state after dividing for the $n\text{th}$ time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a.i) Write down the value of b. [1]

(a.ii) What does b represent in this context? [1]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

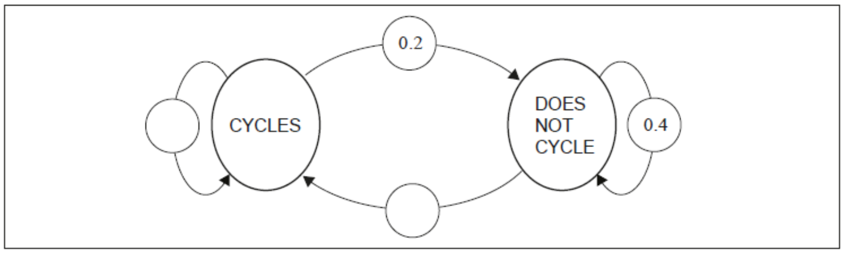
(d.i) when $n = 5$. [2]

(d.ii) in the long term. [2]

5. [Maximum mark: 5] 21N.1.AHL.TZ0.9

Katie likes to cycle to work as much as possible. If Katie cycles to work one day then she has a probability of 0.2 of not cycling to work on the next work day. If she does not cycle to work one day then she has a probability of 0.4 of not cycling to work on the next work day.

(a) Complete the following transition diagram to represent this information.



[2]

(b) Katie works for 180 days in a year.

Find the probability that Katie cycles to work on her final working day of the year. [3]

6. [Maximum mark: 14] SPM.2.AHL.TZ0.6

A city has two cable companies, X and Y. Each year 20 % of the customers using company X move to company Y and 10 % of the customers using company Y move to company X. All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movements between the two companies in a particular year. [2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$. [4]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [2]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

7. [Maximum mark: 8] 21M.1.AHL.TZ2.13

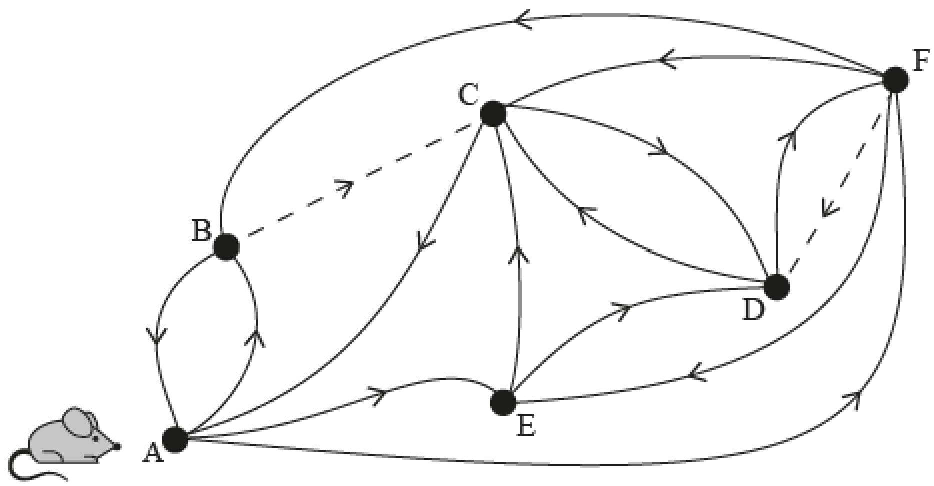
The graph below shows a small maze, in the form of a network of directed routes. The vertices A to F show junctions in the maze and the edges show the possible paths available from one vertex to another.

A mouse is placed at vertex A and left to wander the maze freely. The routes shown by dashed lines indicate paths sprinkled with sugar.

When the mouse reaches any junction, she rests for a constant time before continuing.

At any junction, it may also be assumed that

- the mouse chooses any available normal path with equal probability
- if the junction includes a path sprinkled with sugar, the probability of choosing this path is twice that of a normal path.



(a) Determine the transition matrix for this graph. [3]

(b) If the mouse was left to wander indefinitely, use your graphic display calculator to estimate the percentage of time that the mouse would spend at point F. [3]

(c) Comment on your answer to part (b), referring to at least one limitation of the model. [2]

8. [Maximum mark: 13] 21M.2.AHL.TZ1.5

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}.$$

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c.i) Write down the matrix $\boldsymbol{P}$. [1]

(c.ii) Write down the matrix $\boldsymbol{D}$. [1]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]