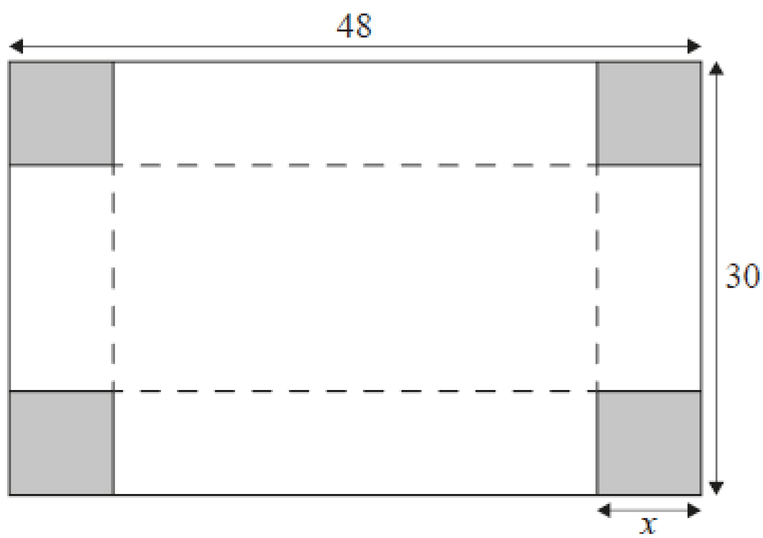
Optimization [87 marks]

1. [Maximum mark: 6] 23M.1.SL.TZ1.7

A rectangular box, with an open top, is to be constructed from a piece of cardboard that measures $48\ \text{cm}$ by $30\ \text{cm}$.

Squares of equal size will be cut from the corners of the cardboard, as indicated by the shading in the diagram. The sides will then be folded along the dotted lines to form the box.

diagram not to scale



The volume of the box, in cubic centimetres, can be modelled by the function $V(x) = (48 - 2x)(30 - 2x)(x)$, for $0 < x < k$, where x is the length of the sides of the squares removed in centimetres.

(a) Write down the maximum possible value of k in this context. [1]

Markscheme

$(k =)\ 15$ ***A1***

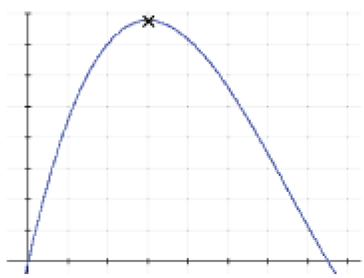
[1 mark]

(b) Find the value of x that maximizes the volume of the box. [2]

Markscheme

EITHER

attempt to sketch the function $V(x)$ with indication of maximum ***(M1)***



OR

recognition of setting the derivative to 0 ***(M1)***

e.g. $V\prime(x) = 0$

THEN

$(x =)\ 6\ (\text{cm})$ ***A1***

Note: Award ***(M1)A0*** for the maximum given as a coordinate pair.

[2 marks]

A second piece of $48\ \text{cm}$ by $30\ \text{cm}$ cardboard is damaged and a strip $2\ \text{cm}$ wide must be removed from all four sides. A box will then be constructed in a similar manner from the remaining cardboard.

(c) Calculate the maximum possible volume of the box made from the second piece of cardboard. [3]

Markscheme

44 or 26 seen ***(A1)***

attempting to adjust the constant(s) in the given volume formula ***(M1)***

volume of second box $= (44 - 2x)(26 - 2x)(x)$

(New maximum volume $=$) $2730\ \mathrm{cm}^3 \quad \left(2726.13\ldots \mathrm{cm}^3\right)$ ***A1***

Note: Units must be seen to award the final ***A1***. Award ***(A1)(M1)A0*** for the maximum given as a coordinate pair.

[3 marks]

2. [Maximum mark: 6] 21N.1.SL.TZ0.12

The surface area of an open box with a volume of 32 cm^3 and a square base with sides of length $x\text{ cm}$ is given by $S(x) = x^2 + \frac{128}{x}$ where $x > 0$.

(a) Find $S\prime(x)$. [3]

Markscheme

$(S(x) =)\ x^2 + 128x^{-1}$ ***(M1)***

Note: Award ***(M1)*** for expressing second term with a negative power. This may be implied by $\frac{1}{x^2}$ seen as part of their answer.

$2x - \frac{128}{x^2}$ **OR** $2x - 128x^{-2}$ ***A1A1***

Note: Award ***A1*** for $2x$ and ***A1*** for $-\frac{128}{x^2}$. The first ***A1*** is for x^2 differentiated correctly and is independent of the ***(M1)***.

[3 marks]

(b.i) Solve $S\prime(x) = 0$. [2]

Markscheme

EITHER

any correct manipulation of $2x - \frac{128}{x^2} = 0$ e.g. $2x^3 - 128 = 0$ ***(M1)***

OR

sketch of graph of $S\prime(x)$ with root indicated *(M1)*

OR

sketch of graph of $S(x)$ with minimum indicated *(M1)*

THEN

$x = 4$ *A1*

Note: Value must be positive. Follow through from their part (a) irrespective of working.

[2 marks]

(b.ii) Interpret your answer to (b)(i) in context. [1]

Markscheme

the value of x that will minimize surface area of the box *A1*

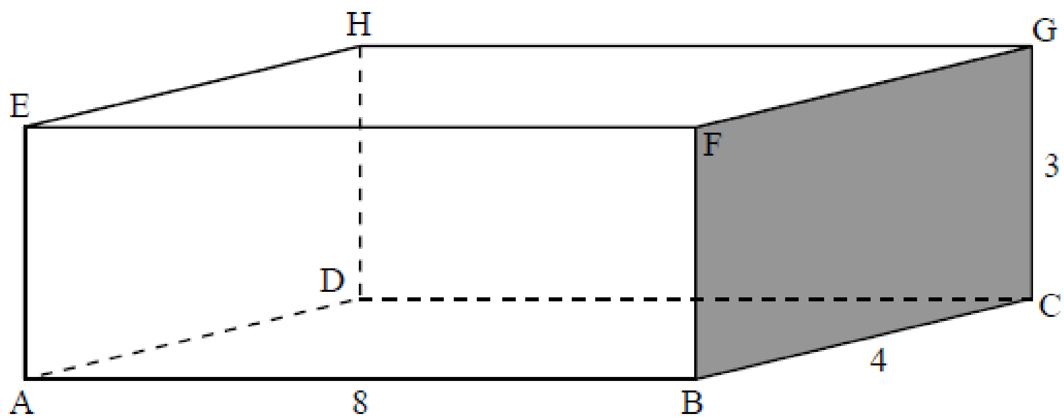
Note: Accept 'optimize' in place of minimize.

[1 mark]

3. [Maximum mark: 15] SPM.2.SL.TZ0.4

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length 8 cm, width 4 cm and height 3 cm. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

Markscheme

$2(8 \times 4 + 3 \times 4 + 3 \times 8)$ ***M1***

$= 136$ (cm^2) ***A1***

[2 marks]

(b) Calculate the length AG. [2]

Markscheme

$\sqrt{8^2 + 4^2 + 3^2}$ ***M1***

$(AG =)\ 9.43$ (cm) $(9.4339\ldots, \sqrt{89})$ ***A1***

[2 marks]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{\mathrm{d}P}{\mathrm{d}x} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

Markscheme

$-2x + 220 = 0$ ***M1***

$x = 110$ ***A1***

110 000 (boxes) ***A1***

[3 marks]

(d) Find $P(x)$. [5]

Markscheme

$P(x) = \int -2x + 220 \,\mathrm{d}x$ ***M1***

Note: Award ***M1*** for evidence of integration.

$P(x) = -x^2 + 220x + c$ ***A1A1***

Note: Award ***A1*** for either $-x^2$ or $220x$ award ***A1*** for both correct terms and constant of integration.

$1700 = -(20)^2 + 220(20) + c$ ***M1***

$c = -2300$

$P(x) = -x^2 + 220x - 2300$ ***A1***

[5 marks]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

Markscheme

$-x^2 + 220x - 2300 = 0$ ***M1***

$x = 11.005$ ***A1***

11 006 (boxes) ***A1***

Note: Award ***M1*** for their $P(x) = 0$, award ***A1*** for their correct solution to x.
Award the final ***A1*** for expressing their solution to the minimum number of boxes. Do not accept 11 005, the nearest integer, nor 11 000, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

[3 marks]

4. [Maximum mark: 12] EXN.2.SL.TZ0.2

A box of chocolates is to have a ribbon tied around it as shown in the diagram below.

The box is in the shape of a cuboid with a height of 3 cm. The length and width of the box are x and y cm.

After going around the box an extra 10 cm of ribbon is needed to form the bow.

(a) Find an expression for the total length of the ribbon L in terms of x and y. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$L = 2x + 2y + 12 + 10 = 2x + 2y + 22$ **A1A1**

Note: A1 for $2x + 2y$ and **A1** for $12 + 10$ or 22.

[2 marks]

The volume of the box is $450\,\mathrm{cm}^3$.

(b) Show that $L = 2x + \frac{300}{x} + 22$ [3]

Markscheme

$V = 3xy = 450$ **A1**

$y = \frac{150}{x}$ **A1**

$L = 2x + 2\left(\frac{150}{x}\right) + 22$ **M1**

$L = 2x + \frac{300}{x} + 22$ **AG**

[3 marks]

(c) Find $\frac{\mathrm{d}L}{\mathrm{d}x}$ [3]

Markscheme

$L = 2x + 300x^{-1} + 22$ **(M1)**

$\frac{\mathrm{d}L}{\mathrm{d}x} = 2 - \frac{300}{x^2}$ **A1A1**

Note: A1 for 2 (and 0), **A1** for $\frac{300}{x^2}$.

[3 marks]

(d) Solve $\frac{\mathrm{d}L}{\mathrm{d}x} = 0$ [2]

Markscheme

$\frac{300}{x^2} = 2$ **(M1)**

$x = \sqrt{150} = 12.2 \ (12.2474\ldots)$ **A1**

[2 marks]

(e) Hence or otherwise find the minimum length of ribbon required. [2]

Markscheme

$L = 2\sqrt{150} + \frac{300}{\sqrt{150}} + 22 = 71.0 \ (70.9897\ldots)\,\mathrm{cm}$

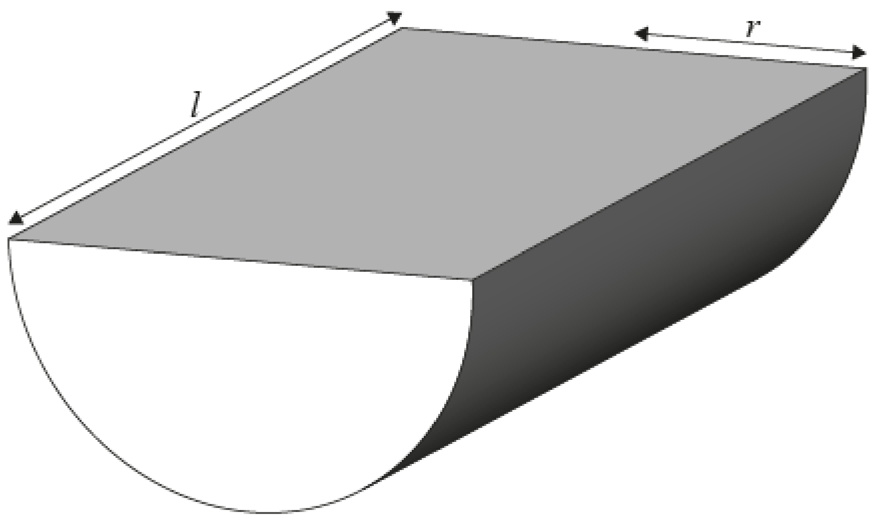
(M1)A1

[2 marks]

5. [Maximum mark: 17] 23M.2.SL.TZ1.5

A large closed container, in the shape of a half cylinder with a rectangular lid, is to be constructed with a volume of $0.8\,\mathrm{m}^3$. The container has a length of l metres and a radius of r metres.

diagram not to scale



(a) Find an exact expression for l in terms of r and π. [2]

Markscheme

equating a volume of a half cylinder (or cylinder) to 0.8 ***(M1)***

$$0.8 = \frac{1}{2}\pi r^2 l$$

$l = \frac{1.6}{\pi r^2}$ ***A1***

Note: Do not accept decimal approximation of π for the ***A1*** given the demand of question.

Condone the use of h for l for the ***M1***

[2 marks]

The container will be constructed using two different materials. The material for both the curved surface and the rectangular lid of the container costs $\$4.40$ per square metre. The material for the semicircular ends of the container costs $\$p$ per square metre.

The cost, C, of the materials to construct the container can be written in terms of r and p (where $p > 0$ and $r > 0$).

(b) Show that $C = 7.04r^{-1} + \frac{14.08}{\pi}r^{-1} + p\pi r^2$. [4]

Markscheme

calculating area in terms of r and l ***M1***

$C = 2lr + \pi r^2 + \pi rl$

area with l replaced by $\frac{1.6}{\pi r^2}$ ***M1***

apply costs to correct part of each surface ***M1***

a correct substitution into an expression for C, leading to given answer ***A1***

e.g. $(C =)\ 4.40 \times \pi r\left(\frac{1.6}{\pi r^2}\right) + 4.40 \times 2r\left(\frac{1.6}{\pi r^2}\right) + p \times \pi r^2$

$(C =)\ 7.04r^{-1} + \frac{14.08}{\pi}r^{-1} + p\pi r^2$ ***AG***

Note: The ***AG*** line must be seen to award the final ***A1***.

No incorrect working should be seen after the correct substitution

[4 marks]

(c) Find $\frac{\mathrm{d}C}{\mathrm{d}r}$. [3]

Markscheme

EITHER

$\left(\frac{\mathrm{d}C}{\mathrm{d}r} =\right) -7.04r^{-2} - \frac{14.08}{\pi}r^{-2} + 2p\pi r$ ***A1A1A1***

OR

$-7.04r^{-2} - 4.48r^{-2} + 6.28pr$
$\left(-7.04r^{-2} - \left(4.48180\ldots\right)r^{-2} + 6.28318\ldots pr\right)$ ***A1A1A1***

OR

$-11.5r^{-2} + 6.28pr$ $\left(\left(-11.5218\ldots\right)r^{-2} + 6.28318\ldots pr\right)$
A2A1

Note: Award ***A1*** for each correct term.

Award at most ***A1A1A0*** if extra terms are seen.

[3 marks]

The cost of materials to construct the container is minimized when the radius of the container, r, is $0.7\,\mathrm{m}$.

(d) Find the value of p. [3]

Markscheme

recognition of setting $\frac{\mathrm{d}C}{\mathrm{d}r}$ to zero ***(M1)***

attempt to substitute 0.7 in for r in their derivative ***(M1)***

$$0 = -7.04(0.7)^{-2} - \frac{14.08}{\pi} \times (0.7)^{-2} + 2p\pi \times 0.7$$

$(p =)$ $(\$)5.35$ (per square metre) $((\$)\ 5.34621\ldots)$ ***A1***

Note: Accept $\$5.34$, as this will also lead to a radius of 0.7 (to 3sf).

[3 marks]

In total, 350 containers will be constructed at this minimum cost.

(e) Calculate the cost of materials, to the nearest dollar, to construct all 350 containers. [3]

Markscheme

attempt to calculate the cost of one container ***(M1)***

$$\left(C =\right) 7.04(0.7)^{-1} + \frac{14.08}{\pi}(0.7)^{-1} + 5.34621\ldots\pi \times 0.7^{2}$$

(A1)

Note: May be shown within a calculation of the cost of all containers.

$(C =)\ 24.6895\ldots$

$24.6895\ldots \times 350$

$= (\$)\ 8641$ ***A1***

Note: Answer must be rounded to the nearest dollar to award the final ***A1***.

Accept answers between 8641 and 8645 (inclusive), due to rounding the value of p and/or the cost of one container to the nearest cent.

Award ***(M1)(A1)A0*** for an answer rounded to 3sf (e.g. $(\$)8640$) or to 2dp (e.g., $(\$)8641.35$).

Accept an answer of $(\$)8638$ from use of $\$5.34$ in their cost calculation.

[3 marks]

The materials for constructing the containers can be purchased at a discount according to the information in the table.

Cost of materials ($\$C$) before discount	**Discount applied to entire order**
$1000 \le C < 2500$	1%
$2500 \le C < 5000$	4%
$5000 \le C < 10\,000$	8%
$C \ge 10\,000$	10%

(f) Determine the cost of materials for 350 containers after the discount is applied.

[2]

Markscheme

attempt to apply a discount of 8% to their part (e) ***(M1)***

Note: the discount percentage will depend on their answer to part (e)

e.g. $8641.35\ldots \times 0.92$ **OR** $8641.35\ldots \times 0.08$

$(\$)7950 \quad ((\$)7950.04\ldots)$ ***A1***

[2 marks]

6. [Maximum mark: 16] 23M.2.SL.TZ2.5

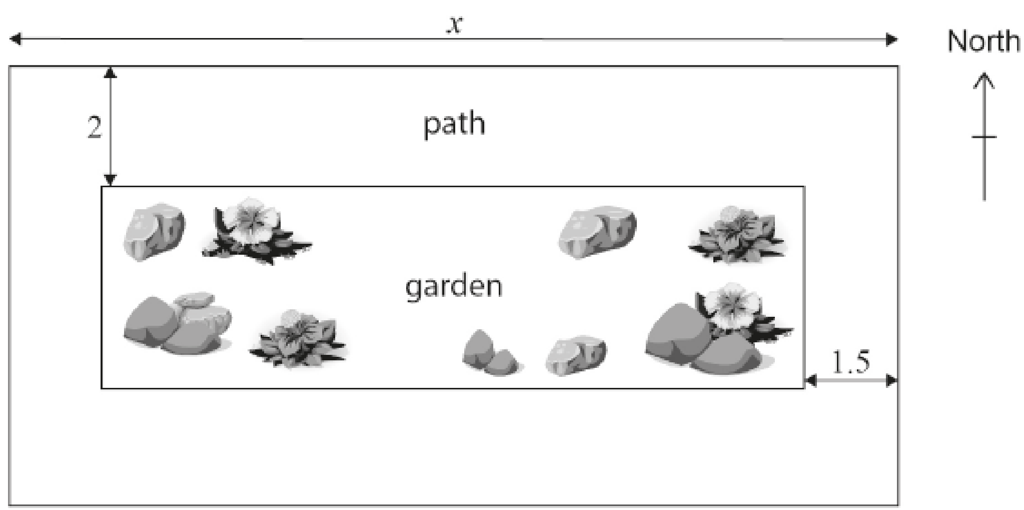
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a.i) Write down the length of the garden in terms of x. [1]

Markscheme

$x - 3$ ***A1***

[1 mark]

(a.ii) Find an expression for the width of the garden in terms of x. [2]

Markscheme

attempt to use 1200 to find width of park in terms of only x ***(M1)***

$\frac{1200}{x}$ (seen) **OR** $1200 = x \times$park width **OR** $1200 = x \times$(garden width $+4$)

$\frac{1200}{x} - 4$ ***A1***

[2 marks]

(a.iii) Hence show that $A = 1212 - 4x - \frac{3600}{x}$. [2]

Markscheme

$A = (x - 3) \times \left(\frac{1200}{x} - 4\right)$ ***A1***

$= 1200 - 4x - \frac{3600}{x} + 12$ ***A1***

Note: Award first ***A1FT*** for multiplying their garden length and width and second ***A1*** for a simplified (parentheses removed) expression for A that leads to the given answer. The given answer must be shown for the second ***A1*** mark to be awarded

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

[2 marks]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

Markscheme

setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) ***(M1)***

$x = 9.64\ (9.64011\ldots)\ (\mathrm{m})$ **OR**
$x = 9.34\ (93.3598\ldots)\ (\mathrm{m})$ ***A1***

(width =) $124\ (124.479\ldots)\ (\mathrm{m})$ ***A1***

(width =) $12.9\ (12.8534\ldots)\ (\mathrm{m})$ ***A1***

Note: To award the final ***A1*** both values of x **and** both values of the width must be seen. Accept 12.8 for second value of width from candidate dividing 1200 by 3 sf value of 93.4.

[4 marks]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

Markscheme

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) - 4 + \frac{3600}{x^2}$ **OR** $-4 + 3600x^{-2}$ ***A1A1A1***

Note: Award ***A1*** for -4, **A1** for $+3600$, and ***A1*** for x^{-2} or x^2 in denominator.

[3 marks]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

Markscheme

setting *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ equal to 0 **OR** sketch of *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ with x-intercept highlighted ***M1***

$(x =)\ 30\ (\mathrm{m})$ ***A1***

Note: To award ***A1FT*** the candidate's value of x must be within the domain given in the problem $(3 < x < 300)$.

[2 marks]

(e) Find the maximum possible area of the garden. [2]

Markscheme

EITHER

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ ***(M1)***

OR

substitution of *their* x into A ***(M1)***

OR

dividing 1200 by *their* x to find width of park and subtracting 3 from *their* x and 4 from the width to find park dimensions ***(M1)***

Note: For the last two methods, only follow through if $3 <$their $x < 300$.

THEN

$(A =)\ 972\ (\mathrm{m}^2)$ ***A1***

[2 marks]

7. [Maximum mark: 15] 22M.2.SL.TZ2.5

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

Markscheme

attempt to expand given expression ***(M1)***

$$C = \frac{xk^2}{10} - \frac{3x^3}{1000}$$

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{k^2}{10} - \frac{9x^2}{1000}$$ ***M1A1***

Note: Award ***M1*** for power rule correctly applied to at least one term and ***A1*** for correct answer.

[3 marks]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

Markscheme

equating their $\frac{\mathrm{d}C}{\mathrm{d}x}$ to zero ***(M1)***

$$\frac{k^2}{10} - \frac{9x^2}{1000} = 0$$

$$x^2 = \frac{100k^2}{9}$$

$x = \frac{10k}{3}$ ***(A1)***

substituting their x back into given expression ***(M1)***

$C_{\max} = \frac{10k}{30}\left(k^2 - \frac{300k^2}{900}\right)$

$C_{\max} = \frac{2k^3}{9}\ (0.222\ldots k^3)$ ***A1***

[4 marks]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c.i) Find the value of k. [2]

Markscheme

substituting 20 into given expression and equating to 426 ***M1***

$426 = \frac{20}{10}\left(k^2 - \frac{3}{100}(20)^2\right)$

$k = 15$ ***A1***

[2 marks]

(c.ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [1]

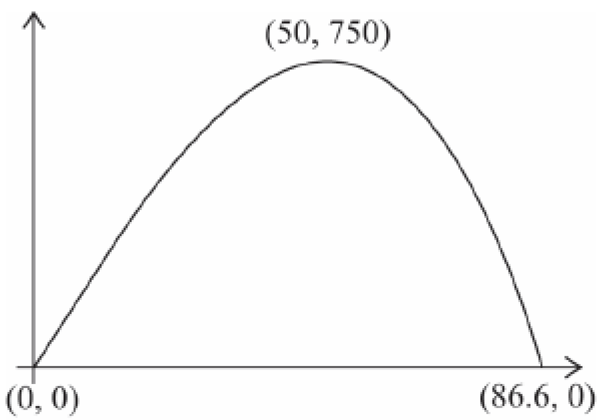
Markscheme

50 ***A1***

[1 mark]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

Markscheme



A1A1A1

Note: Award ***A1*** for graph drawn for positive x indicating an increasing and then decreasing function, ***A1*** for maximum labelled and ***A1*** for graph passing through the origin and 86.6, marked on the x-axis or whose coordinates are given.

[3 marks]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

Markscheme

setting their expression for C to zero **OR** choosing correct x-intercept on their graph of C ***(M1)***

$x_{\max} = 86.6\ \ (86.6025\ldots)$ litres ***A1***

[2 marks]

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