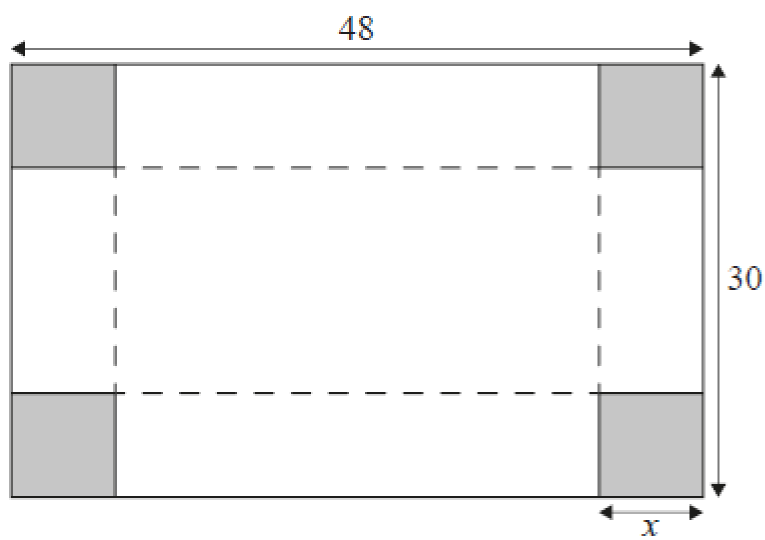
Optimization [87 marks]

1. [Maximum mark: 6] 23M.1.SL.TZ1.7

A rectangular box, with an open top, is to be constructed from a piece of cardboard that measures $48\ \mathrm{cm}$ by $30\ \mathrm{cm}$.

Squares of equal size will be cut from the corners of the cardboard, as indicated by the shading in the diagram. The sides will then be folded along the dotted lines to form the box.

diagram not to scale



The volume of the box, in cubic centimetres, can be modelled by the function $V(x) = (48 - 2x)(30 - 2x)(x)$, for $0 < x < k$, where x is the length of the sides of the squares removed in centimetres.

(a) Write down the maximum possible value of k in this context. [1]

(b) Find the value of x that maximizes the volume of the box. [2]

A second piece of $48\ \mathrm{cm}$ by $30\ \mathrm{cm}$ cardboard is damaged and a strip $2\ \mathrm{cm}$ wide must be removed from all four sides. A box will then be constructed in a similar manner from the remaining cardboard.

(c) Calculate the maximum possible volume of the box made from the second piece of cardboard.

[3]

2. [Maximum mark: 6] 21N.1.SL.TZ0.12

The surface area of an open box with a volume of $32\,\mathrm{cm}^3$ and a square base with sides of length $x\,\mathrm{cm}$ is given by $S(x) = x^2 + \frac{128}{x}$ where $x > 0$.

(a) Find $S'(x)$. [3]

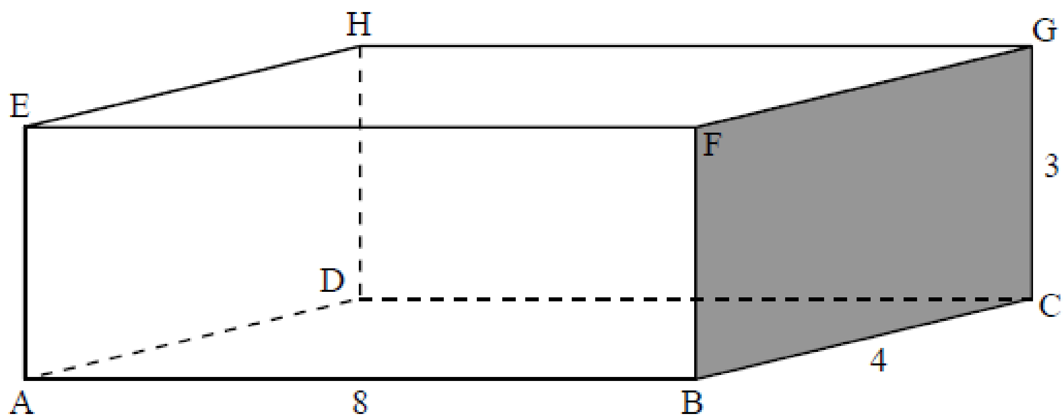
(b.i) Solve $S'(x) = 0$. [2]

(b.ii) Interpret your answer to (b)(i) in context. [1]

3. [Maximum mark: 15] SPM.2.SL.TZ0.4

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length 8 cm, width 4 cm and height 3 cm. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{\mathrm{d}P}{\mathrm{d}x} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

(d) Find $P(x)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

4. [Maximum mark: 12] EXN.2.SL.TZ0.2

A box of chocolates is to have a ribbon tied around it as shown in the diagram below.

The box is in the shape of a cuboid with a height of 3 cm. The length and width of the box are x and y cm.

After going around the box an extra 10 cm of ribbon is needed to form the bow.

(a) Find an expression for the total length of the ribbon L in terms of x and y. [2]

The volume of the box is $450\,\mathrm{cm}^3$.

(b) Show that $L = 2x + \frac{300}{x} + 22$ [3]

(c) Find $\frac{\mathrm{d}L}{\mathrm{d}x}$ [3]

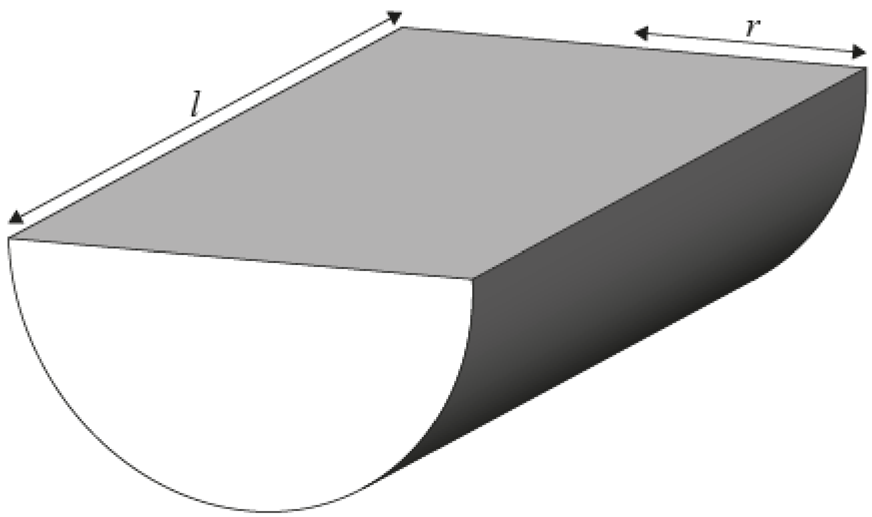
(d) Solve $\frac{\mathrm{d}L}{\mathrm{d}x} = 0$ [2]

(e) Hence or otherwise find the minimum length of ribbon required. [2]

5. [Maximum mark: 17] 23M.2.SL.TZ1.5

A large closed container, in the shape of a half cylinder with a rectangular lid, is to be constructed with a volume of $0.8\,\mathrm{m}^3$. The container has a length of l metres and a radius of r metres.

diagram not to scale



(a) Find an exact expression for l in terms of r and π. [2]

The container will be constructed using two different materials. The material for both the curved surface and the rectangular lid of the container costs $\$4.40$ per square metre. The material for the semicircular ends of the container costs $\$p$ per square metre.

The cost, C, of the materials to construct the container can be written in terms of r and p (where $p > 0$ and $r > 0$).

(b) Show that $C = 7.04r^{-1} + \frac{14.08}{\pi}r^{-1} + p\pi r^2$. [4]

(c) Find $\frac{\mathrm{d}C}{\mathrm{d}r}$. [3]

The cost of materials to construct the container is minimized when the radius of the container, r, is $0.7\,\mathrm{m}$.

(d) Find the value of p. [3]

In total, 350 containers will be constructed at this minimum cost.

(e) Calculate the cost of materials, to the nearest dollar, to construct all 350 containers. [3]

The materials for constructing the containers can be purchased at a discount according to the information in the table.

Cost of materials ($\$C$) before discount	**Discount applied to entire order**
$1000 \leq C < 2500$	1%
$2500 \leq C < 5000$	4%
$5000 \leq C < 10\,000$	8%
$C \geq 10\,000$	10%

(f) Determine the cost of materials for 350 containers after the discount is applied. [2]

6. [Maximum mark: 16] 23M.2.SL.TZ2.5

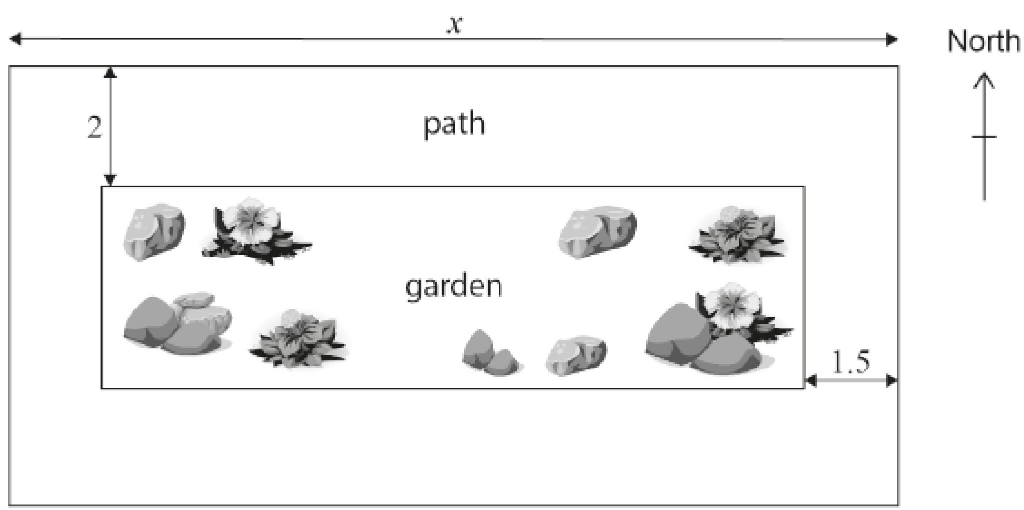
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a.i) Write down the length of the garden in terms of x. [1]

(a.ii) Find an expression for the width of the garden in terms of x. [2]

(a.iii) Hence show that $A = 1212 - 4x - \frac{3600}{x}$. [2]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden.

[2]

(e) Find the maximum possible area of the garden. [2]

7. [Maximum mark: 15] 22M.2.SL.TZ2.5

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c.i) Find the value of k. [2]

(c.ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [1]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]