Transformations [86 marks]

1. [Maximum mark: 16] EXM.2.AHL.TZ0.16

The matrices A and B are defined by $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by B. [2]

Markscheme

reflection in the y-axis ***A1A1***

[2 marks]

Triangle X is mapped onto triangle Y by the transformation represented by AB. The coordinates of triangle Y are (0, 0), (−30, −20) and (−16, 32).

(b) Find the coordinates of triangle X. [5]

Markscheme

$X = (AB)^{-1}Y$ ***M1***

EITHER

$AB = \begin{pmatrix} -3 & -2 \\ -2 & 4 \end{pmatrix}$, so $(AB)^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{16} \end{pmatrix}$ ***M1A1***

OR

$X = B^{-1}A^{-1}Y$ ***M1A1***

THEN

$X = \begin{pmatrix} 0 & 10 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ ***(A1)***

So the coordinates are (0, 0), (10, 0) and (0, 8). ***A1***

[5 marks]

(c.i) Find the area of triangle X. [2]

Markscheme

$\frac{10\times8}{2} = 40$ units^2 ***M1A1***

[2 marks]

(c.ii) Hence find the area of triangle Y. [3]

Markscheme

$\det\ (AB) = -16$ ***M1A1***

Area $= 40 \times 16 = 640$ units^2 ***A1***

[3 marks]

(d) Matrix A represents a combination of transformations:

A stretch, with scale factor 3 and y-axis invariant;
Followed by a stretch, with scale factor 4 and x-axis invariant;
Followed by a transformation represented by matrix C.

Find matrix C. [4]

Markscheme

A stretch, with scale factor 3 and y-axis invariant is given by $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

A stretch, with scale factor 4 and x-axis invariant is given by $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ ***A1***

So $C = A\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}^{-1} = \begin{pmatrix}1 & -\frac{1}{2}\\\frac{2}{3} & 1\end{pmatrix}$ ***M1A1***

[4 marks]

2. [Maximum mark: 12] EXM.2.AHL.TZ0.15

The matrix A is defined by $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by A. [5]

Markscheme

stretch ***A1***

scale factor 3, ***A1***

y-axis invariant (condone parallel to the x-axis) ***A1***

and

stretch, scale factor 2, ***A1***

x-axis invariant (condone parallel to the y-axis) ***A1***

[5 marks]

Pentagon, P, which has an area of 7 cm^2, is transformed by A.

(b) Find the area of the image of P. [2]

Markscheme

$\det(A) = 6$ ***A1***

$7 \times 6 = 42\,\text{cm}^2$ ***A1***

[2 marks]

The matrix B is defined by $B = \frac{1}{2}\begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$.

B represents the combined effect of the transformation represented by a matrix X, followed by the transformation represented by A.

(c) Find the matrix X. [3]

Markscheme

$B = AX$ ***(A1)***

$X = A^{-1}B$ ***(M1)***

$$X = \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix} \left(= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \right)$$ ***A1***

[3 marks]

(d) Describe fully the geometrical transformation represented by X. [2]

Markscheme

Rotation ***A1***

clockwise by 30° about the origin ***A1***

[2 marks]

3. [Maximum mark: 7] 23M.1.AHL.TZ2.13

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, $T\prime$.

(a) Find the single matrix that represents the transformation $T\prime \rightarrow T$, which will undo the transformation described above. [4]

Markscheme

METHOD 1 (find product of matrices first)

$T \rightarrow T\prime$ is represented by $\boldsymbol{QP} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} -12 & -2 \\ 3 & 7 \end{pmatrix}$ ***(A1)***

recognizing need to find their $(\boldsymbol{QP})^{-1}$ ***(M1)***

$(\boldsymbol{QP})^{-1} = \begin{pmatrix} -12 & -2 \\ 3 & 7 \end{pmatrix}^{-1}$

$= \frac{1}{78}\begin{pmatrix} 7 & 2 \\ -3 & -12 \end{pmatrix}$ **OR**

$= \begin{pmatrix} -0.0897435\ldots & -0.0256410\ldots \\ 0.0384615\ldots & 0.153846\ldots \end{pmatrix}$ ***A1***

METHOD 2 (find inverses of both matrices first)

recognizing need to find inverse of both $\boldsymbol{P}$ and $\boldsymbol{Q}$ ***(M1)***

$$\boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix}$$ **AND** $$\boldsymbol{Q}^{-1} = \begin{pmatrix} -\frac{3}{13} & \frac{1}{13} \\ \frac{1}{13} & \frac{4}{13} \end{pmatrix}$$ ***(A1)***

$T\prime \to T$ is represented by $\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$

(M1)

$= \frac{1}{78}\begin{pmatrix} 7 & 2 \\ -3 & -12 \end{pmatrix}$ **OR**

$= \begin{pmatrix} -0.0897435\ldots & -0.0256410\ldots \\ 0.0384615\ldots & 0.153846\ldots \end{pmatrix}$ ***A1***

Note: In METHOD 1, award ***M1A0M1A0*** if they multiply the matrices in the wrong order.

In METHOD 2, award ***M1A1M1A0*** if they multiply the matrices in the wrong order.

[4 marks]

The area of $T\prime$ is $273\ \text{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

Markscheme

$\left(\det\left[-\frac{1}{78}\begin{pmatrix} 7 & 2 \\ -3 & -12 \end{pmatrix}\right] =\right) - \frac{1}{78}$ **OR**

$\left(\det\begin{pmatrix} -12 & -2 \\ 3 & 7 \end{pmatrix} =\right) - 78$ ***(A1)*** area of $T\prime = |\det \boldsymbol{QP}| \times \text{area of } T$ **OR** area of $T = \left|\det (\boldsymbol{QP})^{-1}\right| \times \text{area of } T\prime$ ***(M1)***

$\Rightarrow$area of $T = 273 \times \frac{1}{78}$

$= 3.5\ (\text{cm}^2)$ ***A1***

Note: Award ***(A1)(M0)A0*** for an answer of $-3.5\ (\text{cm}^2)$ with or without working. Accept an answer of $4.04\ (\text{cm}^2)$ from use of 3sf values in their answer to part (a).

[3 marks]

4. [Maximum mark: 8] 22N.1.AHL.TZ0.9

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Markscheme

attempt to find $\det(\boldsymbol{M})$ ***(M1)***

$= 14$

$(12 \times 14) = 168\,\text{cm}^2$ ***A1***

[2 marks]

Under the transformation T, the image of point X has coordinates $(2t - 3,\ 6 - 5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]

Markscheme

let X have coordinates $(x,\ y)$

METHOD 1

$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t - 3 \\ 6 - 5t \end{pmatrix}$ ***(M1)***

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{M}^{-1} \begin{pmatrix} 2t-3 \\ 6-5t \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{M}^{-1} = \frac{1}{14} \begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 2t-3+24-20t \\ -6t+9+12-10t \end{pmatrix}$$ ***(M1)***

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 21-18t \\ 21-16t \end{pmatrix}$$ **OR** $\left(\frac{21-18t}{14},\ \frac{21-16t}{14}\right)$ ***A1A1***

METHOD 2

writing two simultaneous equations ***(M1)***

$2x - 4y = 2t - 3$ ***(A1)***

$3x + y = 6 - 5t$ ***(A1)***

attempting to solve the equations ***(M1)***

$(x,\ y) = \left(\frac{3}{2} - \frac{9t}{7},\ \frac{3}{2} - \frac{8t}{7}\right)$ ***A1A1***

[6 marks]

5. [Maximum mark: 18] 22M.2.AHL.TZ1.7

A transformation, T, of a plane is represented by $\boldsymbol{r}\prime = \boldsymbol{P}\boldsymbol{r} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}\prime$ the position vector of its image under T.

The triangle OAB has coordinates $(0,\ 0)$, $(0,\ 1)$ and $(1,\ 0)$. Under T, these points are transformed to $(0,\ 1)$, $\left(\frac{1}{4},\ 1+\frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4},\ \frac{3}{4}\right)$ respectively.

(a.i) By considering the image of $(0,\ 0)$, find $\boldsymbol{q}$. [2]

Markscheme

$$\boldsymbol{P}\begin{pmatrix}0\\0\end{pmatrix} + \boldsymbol{q} = \begin{pmatrix}0\\1\end{pmatrix}$$ ***(M1)***

$$\boldsymbol{q} = \begin{pmatrix}0\\1\end{pmatrix}$$ ***A1***

[2 marks]

(a.ii) By considering the image of $(1,\ 0)$ and $(0,\ 1)$, show that

$$\boldsymbol{P} = \begin{pmatrix}\frac{\sqrt{3}}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{\sqrt{3}}{4}\end{pmatrix}.$$

[4]

Markscheme

EITHER

$$\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{4}\\ \frac{3}{4}\end{pmatrix}$$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$ ***A1***

$\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\1+\frac{\sqrt{3}}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$ ***A1***

OR

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\\frac{3}{4}\end{pmatrix}$ ***M1***

hence $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$ ***A1***

$\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\1+\frac{\sqrt{3}}{4}\end{pmatrix}$ ***M1***

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$ ***A1***

$\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$

THEN

$$\Rightarrow \boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***AG***

[4 marks]

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{RS}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0,\ 0)$.

$\boldsymbol{R}$ represents a rotation about $(0,\ 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

Markscheme

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$ ***A1***

[1 mark]

(c.i) Use $\boldsymbol{P} = \boldsymbol{RS}$ to find the matrix $\boldsymbol{R}$. [4]

Markscheme

EITHER

$$\boldsymbol{S}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{R} = \boldsymbol{PS}^{-1}$$ ***(M1)***

Note: The ***M1*** is for an attempt at rearranging the matrix equation. Award even if the order of the product is reversed.

$$\boldsymbol{R} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ ***(A1)***

OR

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

let $\boldsymbol{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

attempt to solve a system of equations ***M1***

$$\frac{\sqrt{3}}{4} = 0.5a, \quad \frac{1}{4} = 0.5b$$

$$-\frac{1}{4} = 0.5c, \quad \frac{\sqrt{3}}{4} = 0.5d$$ ***A2***

Note: Award ***A1*** for two correct equations, ***A2*** for all four equations correct.

THEN

$$\boldsymbol{R} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$ **OR** $$\begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix}$$ **OR**

$$\left(\begin{pmatrix} 0.866025\ldots & 0.5 \\ -0.5 & 0.866025\ldots \end{pmatrix}\right)$$ ***A1***

Note: The correct answer can be obtained from reversing the matrices, so do not award if incorrect product seen. If the given answer is obtained from the product $\boldsymbol{R} = \boldsymbol{S}^{-1}\boldsymbol{P}$, award ***(A1)(M1)(A0)A0***.

[4 marks]

(c.ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [3]

Markscheme

clockwise ***A1***

arccosine or arcsine of value in matrix seen ***(M1)***

30° ***A1***

Note: Both ***A1*** marks are dependent on the answer to part (c)(i) and should only be awarded for a valid rotation matrix.

[3 marks]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre $(a,\ b)$, followed by a rotation about the same centre $(a,\ b)$.

(d.i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$. [1]

Markscheme

METHOD 1

$$\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$$ ***A1***

METHOD 2

$$\begin{pmatrix} x\prime \\ y\prime \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} x - a \\ y - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$ ***A1***

Note: Accept substitution of x and y (and $x\prime$ and $y\prime$) with particular points given in the question.

[1 mark]

(d.ii) Find the value of a and the value of b. [3]

Markscheme

METHOD 1

solving $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ using simultaneous equations or

$\boldsymbol{a} = (\boldsymbol{I} - \boldsymbol{P})^{-1}\boldsymbol{q}$ ***(M1)***

$a = 0.651\ \ (0.651084\ldots),\ \ b = 1.48\ \ (1.47662\ldots)$ ***A1A1***

$\left(a = \frac{5+2\sqrt{3}}{13},\ b = \frac{14+3\sqrt{3}}{13}\right)$

METHOD 2

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} 0 - a \\ 0 - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$ ***(M1)***

Note: This line, with any of the points substituted, may be seen in part (d)(i) and if so the ***M1*** can be awarded there.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{P})\begin{pmatrix} a \\ b \end{pmatrix}$$

$a = 0.651084\ldots,\ \ b = 1.47662\ldots$ ***A1A1***

$\left(a = \frac{5+2\sqrt{3}}{13},\ b = \frac{14+3\sqrt{3}}{13}\right)$

[3 marks]

6. [Maximum mark: 18] 21N.2.AHL.TZ0.4

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a.i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation. [6]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

rotation anticlockwise $\frac{\pi}{6}$ is $\begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$ OR $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

(M1)A1

reflection in $y = \frac{x}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ ***(M1)***

$\Rightarrow 2\theta = \frac{\pi}{3}$ ***(A1)*** matrix is $\begin{pmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ ***A1***

rotation clockwise $\frac{\pi}{3}$ is $\begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

A1

[6 marks]

(a.ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position. [3]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

an attempt to multiply three matrices ***(M1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \textbf{ OR } \begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}$$ ***A1***

[3 marks]

(a.iii) Find $\boldsymbol{P}^2$. [1]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

$$\left(\boldsymbol{P}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} =\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ***A1***

Note: Do not award ***A1*** if final answer not resolved into the identity matrix I.

[1 mark]

(a.iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [2]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

if the overall movement of the drone is repeated ***A1***

the drone would return to its original position ***A1***

[2 marks]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points $\mathrm{A}\prime$, $\mathrm{B}\prime$ and $\mathrm{C}\prime$ respectively.

Show that the area of triangle ABC is equal to the area of triangle $\mathrm{A}\prime\mathrm{B}\prime\mathrm{C}\prime$. [2]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

METHOD 1

$|\det \boldsymbol{P}| = \left|\left(-\frac{3}{4}\right) - \left(\frac{1}{4}\right)\right| = 1$ ***A1***

area of triangle $\mathrm{ABC} =$ area of triangle $\mathrm{A}\prime\mathrm{B}\prime\mathrm{C}\prime \times |\det \boldsymbol{P}|$ ***R1***

area of triangle $\mathrm{ABC} =$ area of triangle $\mathrm{A}\prime\mathrm{B}\prime\mathrm{C}\prime$ ***AG***

Note: Award at most ***A1R0*** for responses that omit modulus sign.

METHOD 2

statement of fact that rotation leaves area unchanged ***R1*** statement of fact that reflection leaves area unchanged ***R1***

area of triangle ABC = area of triangle $\mathrm{A'B'C'}$ ***AG***

[2 marks]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

Markscheme

Note: For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

attempt to find angles associated with values of elements in matrix $\boldsymbol{P}$ ***(M1)***

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & -\cos\left(-\frac{\pi}{6}\right) \end{pmatrix}$$

reflection (in $y = (\tan\,\theta)x$) ***(M1)***

where $2\theta = -\frac{\pi}{6}$ ***A1***

reflection in $y = \tan\left(-\frac{\pi}{12}\right)x \ \ (= -0.268x)$ ***A1***

[4 marks]

7. [Maximum mark: 7] 21M.1.AHL.TZ2.14

A geometric transformation $T:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\prime\\y\prime\end{pmatrix}$ is defined by

$$T:\begin{pmatrix}x\prime\\y\prime\end{pmatrix}=\begin{pmatrix}7 & -10\\2 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-5\\4\end{pmatrix}.$$

(a) Find the coordinates of the image of the point $(6,\ -2)$. [2]

Markscheme

$$\begin{pmatrix}7 & -10\\2 & -3\end{pmatrix}\begin{pmatrix}6\\-2\end{pmatrix}+\begin{pmatrix}-5\\4\end{pmatrix}$$ ***(M1)***

$=\begin{pmatrix}57\\22\end{pmatrix}$ **OR** $(57,\ 22)$ ***A1***

[2 marks]

(b) Given that $T:\begin{pmatrix}p\\q\end{pmatrix}\mapsto 2\begin{pmatrix}p\\q\end{pmatrix}$, find the value of p and the value of q. [3]

Markscheme

$$\begin{pmatrix}2p\\2q\end{pmatrix}=\begin{pmatrix}7 & -10\\2 & -3\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}+\begin{pmatrix}-5\\4\end{pmatrix}$$ ***(M1)***

$7p-10q-5=2p$

$2p-3q+4=2q$ ***(A1)***

solve simultaneously:

$p=13,\ q=6$ ***A1***

Note: Award ***A0*** if 13 and 6 are not labelled or are labelled the other way around.

[3 marks]

(c) A triangle L with vertices lying on the xy plane is transformed by T.

Explain why both L and its image will have exactly the same area. [2]

Markscheme

$$\det\begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} = -1 \ \left(\textbf{OR} \left|\det\begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix}\right| = 1\right)$$

A1

scale factor of image area is therefore $(|-1| =)1$ (and the translation does not affect the area) ***A1***

[2 marks]