Transformations [86 marks]

1. [Maximum mark: 16] EXM.2.AHL.TZ0.16

The matrices A and B are defined by $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by B. [2]

Triangle X is mapped onto triangle Y by the transformation represented by AB.
The coordinates of triangle Y are (0, 0), (−30, −20) and (−16, 32).

(b) Find the coordinates of triangle X. [5]

(c.i) Find the area of triangle X. [2]

(c.ii) Hence find the area of triangle Y. [3]

(d) Matrix A represents a combination of transformations:

A stretch, with scale factor 3 and y-axis invariant;
Followed by a stretch, with scale factor 4 and x-axis invariant;
Followed by a transformation represented by matrix C.

Find matrix C. [4]

2. [Maximum mark: 12] EXM.2.AHL.TZ0.15

The matrix A is defined by $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by A. [5]

Pentagon, P, which has an area of 7 cm^2, is transformed by A.

(b) Find the area of the image of P. [2]

The matrix B is defined by $B = \frac{1}{2}\begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$.

B represents the combined effect of the transformation represented by a matrix X, followed by the transformation represented by A.

(c) Find the matrix X. [3]

(d) Describe fully the geometrical transformation represented by X. [2]

3. [Maximum mark: 7] 23M.1.AHL.TZ2.13

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, $T\prime$.

(a) Find the single matrix that represents the transformation $T\prime \rightarrow T$, which will undo the transformation described above. [4]

The area of $T\prime$ is 273 cm^2.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

4. [Maximum mark: 8] 22N.1.AHL.TZ0.9

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Under the transformation T, the image of point X has coordinates $(2t - 3,\ 6 - 5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]

5. [Maximum mark: 18] 22M.2.AHL.TZ1.7

A transformation, T, of a plane is represented by $\boldsymbol{r}\prime = \boldsymbol{Pr} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}\prime$ the position vector of its image under T.

The triangle OAB has coordinates $(0,\ 0)$, $(0,\ 1)$ and $(1,\ 0)$. Under T, these points are transformed to $(0,\ 1)$, $\left(\frac{1}{4},\ 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4},\ \frac{3}{4}\right)$ respectively.

(a.i) By considering the image of $(0,\ 0)$, find $\boldsymbol{q}$. [2]

(a.ii) By considering the image of $(1,\ 0)$ and $(0,\ 1)$, show that

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

[4]

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{RS}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0,\ 0)$.

$\boldsymbol{R}$ represents a rotation about $(0,\ 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c.i) Use $\boldsymbol{P} = \boldsymbol{RS}$ to find the matrix $\boldsymbol{R}$. [4]

(c.ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [3]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre $(a,\ b)$, followed by a rotation about the same centre $(a,\ b)$.

(d.i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$. [1]

(d.ii) Find the value of a and the value of b. [3]

6. [Maximum mark: 18] 21N.2.AHL.TZ0.4

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a.i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation. [6]

(a.ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position. [3]

(a.iii) Find $\boldsymbol{P}^2$. [1]

(a.iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [2]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A', B' and C' respectively.

Show that the area of triangle ABC is equal to the area of triangle $\mathrm{A}'\mathrm{B}'\mathrm{C}'$. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

7. [Maximum mark: 7] 21M.1.AHL.TZ2.14

A geometric transformation $T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x\prime \\ y\prime \end{pmatrix}$ is defined by

$$T : \begin{pmatrix} x\prime \\ y\prime \end{pmatrix} = \begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

(a) Find the coordinates of the image of the point $(6,\ -2)$. [2]

(b) Given that $T : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto 2\begin{pmatrix} p \\ q \end{pmatrix}$, find the value of p and the value of q. [3]

(c) A triangle L with vertices lying on the xy plane is transformed by T.

Explain why both L and its image will have exactly the same area. [2]