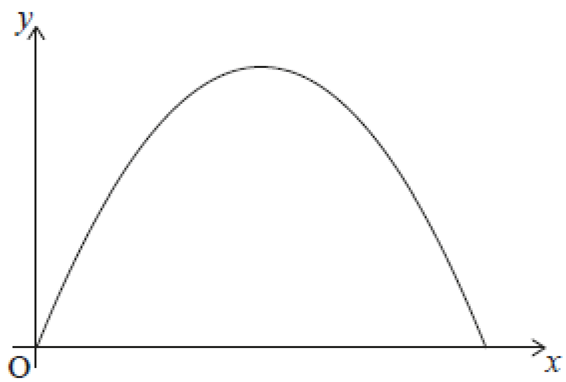
Trig and quadratic modelling [30 marks]

1. [Maximum mark: 5] 22M.1.SL.TZ2.12

The cross-section of an arched entrance into the ballroom of a hotel is in the shape of a parabola. This cross-section can be modelled by part of the graph $y = -1.6x^2 + 4.48x$, where y is the height of the archway, in metres, at a horizontal distance, x metres, from the point O, in the bottom corner of the archway.



(a) Determine the maximum height of the archway. [2]

Markscheme

$(x =)\ -\frac{4.48}{2(-1.6)}$ **OR** coordinates of maximum point $(1.4,\ 3.136)$

(M1)

$x = 1.4$ ***A1***

[2 marks]

To prepare for an event, a square-based crate that is $1.6\,\mathrm{m}$ wide and $2.0\,\mathrm{m}$ high is to be moved through the archway into the ballroom. The crate must remain upright while it is being moved.

(b) Determine whether the crate will fit through the archway.
Justify your answer.

[3]

Markscheme

METHOD 1

the cart is centred in the archway when it is between

$x = 0.6$ and $x = 2.2$, ***A1***

where $y \geq 2.112\,(\text{m})$ (which is greater than 2) ***R1***

the archway is tall enough for the crate ***A1***

Note: Do not award ***R0A1***.

METHOD 2

the height of the archway is greater or equal to 2.0 between

$x = 0.557385\ldots$ and $x = 2.24261\ldots$ ***A1***

width of this section of archway =

$(2.24261\ldots - 0.557385\ldots =)\ \ 1.68522\,(\text{m})$ (which is greater than 1.6) ***R1***

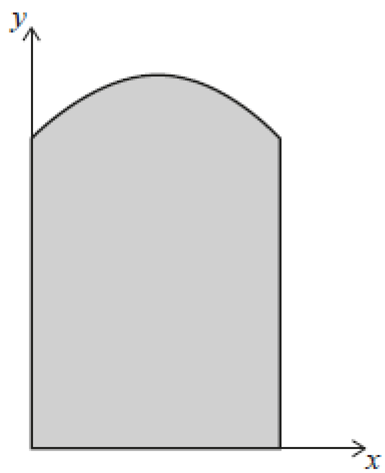
the archway is wide enough for the crate ***A1***

Note: Do not award ***R0A1***.

[3 marks]

2. [Maximum mark: 5] 21N.1.SL.TZ0.13

Irina uses a set of coordinate axes to draw her design of a window. The base of the window is on the x-axis, the upper part of the window is in the form of a quadratic curve and the sides are vertical lines, as shown on the diagram. The curve has end points $(0,\ 10)$ and $(8,\ 10)$ and its vertex is $(4,\ 12)$. Distances are measured in centimetres.



The quadratic curve can be expressed in the form $y = ax^2 + bx + c$ for $0 \leq x \leq 8$.

(a.i) Write down the value of c. [1]

Markscheme

$c = 10$ ***A1***

[1 mark]

(a.ii) Hence form two equations in terms of a and b. [2]

Markscheme

$64a + 8b + 10 = 10$ ***A1***

$16a + 4b + 10 = 12$ ***A1***

Note: Award ***A1*** for each equivalent expression or ***A1*** for the use of the axis of symmetry formula to find $4 = \frac{-b}{2a}$ or from use of derivative. Award ***A0A1*** for $64a + 8b + c = 10$ and $16a + 4b + c = 12$.

[2 marks]

(a.iii) Hence find the equation of the quadratic curve. [2]

Markscheme

$y = -\frac{1}{8}x^2 + x + 10$ ***A1A1***

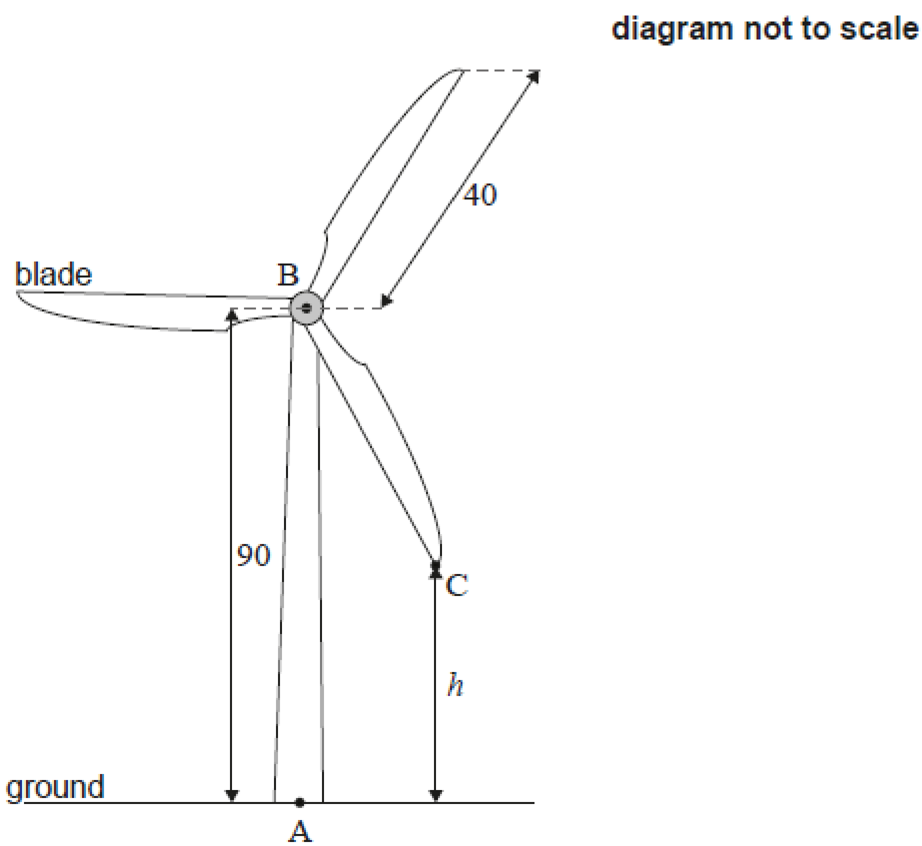
Note: Award ***A1A0*** if one term is incorrect, ***A0A0*** if two or more terms are incorrect. Award at most ***A1A0*** if correct $a,\ b$ and c values are seen but answer not expressed as an equation.

[2 marks]

3. [Maximum mark: 20] 21N.2.SL.TZ0.3

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB, is $90\,\mathrm{m}$. The blades of the turbine are centred at B and are each of length $40\,\mathrm{m}$. This is shown in the following diagram.



The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

Find the

(a.i) maximum value of h. [1]

Markscheme

maximum $h = 130$ metres ***A1***

[1 mark]

(a.ii) minimum value of h. [1]

Markscheme

minimum $h = 50$ metres ***A1***

[1 mark]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b.i) Find the time, in seconds, it takes for the blade $[\mathrm{BC}]$ to make one complete rotation under these conditions. [1]

Markscheme

$(60 \div 12 =)\ 5 \text{ seconds}$ ***A1***

[1 mark]

(b.ii) Calculate the angle, in degrees, that the blade $[\mathrm{BC}]$ turns through in one second. [2]

Markscheme

$360 \div 5$ ***(M1)***

Note: Award ***(M1)*** for 360 divided by their time for one revolution.

$= 72^\circ$ ***A1***

[2 marks]

The height, h, of point C can be modelled by the following function. Time, t, is measured from the instant when the blade $[\mathrm{BC}]$ first passes $[\mathrm{AB}]$ and is measured in seconds.

$$h(t) = 90 - 40\cos(72t^\circ),\ t \geq 0$$

(c.i) Write down the amplitude of the function. [1]

Markscheme

(amplitude =) 40 ***A1***

[1 mark]

(c.ii) Find the period of the function. [1]

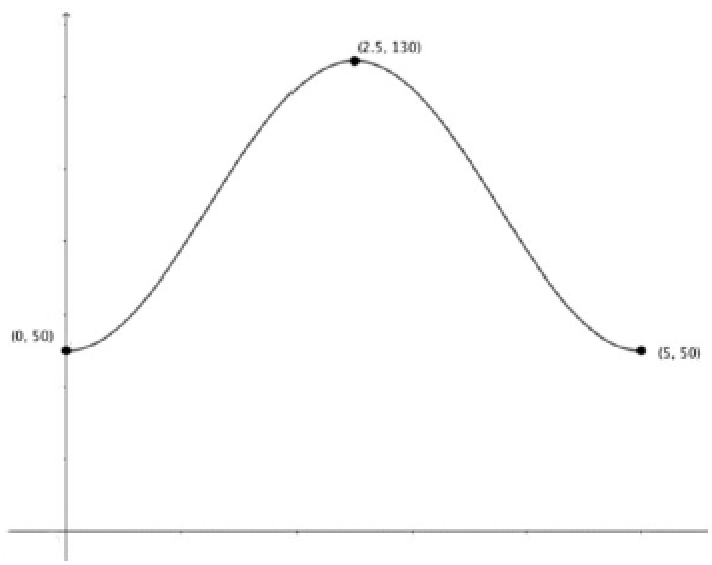
Markscheme

(period $= \frac{360}{72} =$) 5 ***A1***

[1 mark]

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

Markscheme



Maximum point labelled with correct coordinates. ***A1***

At least one minimum point labelled. Coordinates seen for any minimum points must be correct. ***A1***

Correct shape with an attempt at symmetry and "concave up" evident as it approaches the minimum points. Graph must be drawn in the given domain. ***A1***

[3 marks]

(e.i) Find the height of C above the ground when $t = 2$. [2]

Markscheme

$h = 90 - 40\cos(144^\circ)$ ***(M1)***

$(h =)\ 122\,\text{m}\ \ (122.3606\ldots)$ ***A1***

[2 marks]

(e.ii) Find the time, in seconds, that point C is above a height of $100\ \text{m}$, during each complete rotation. [3]

Markscheme

evidence of $h = 100$ on graph **OR** $100 = 90 - 40\cos(72t)$ ***(M1)***

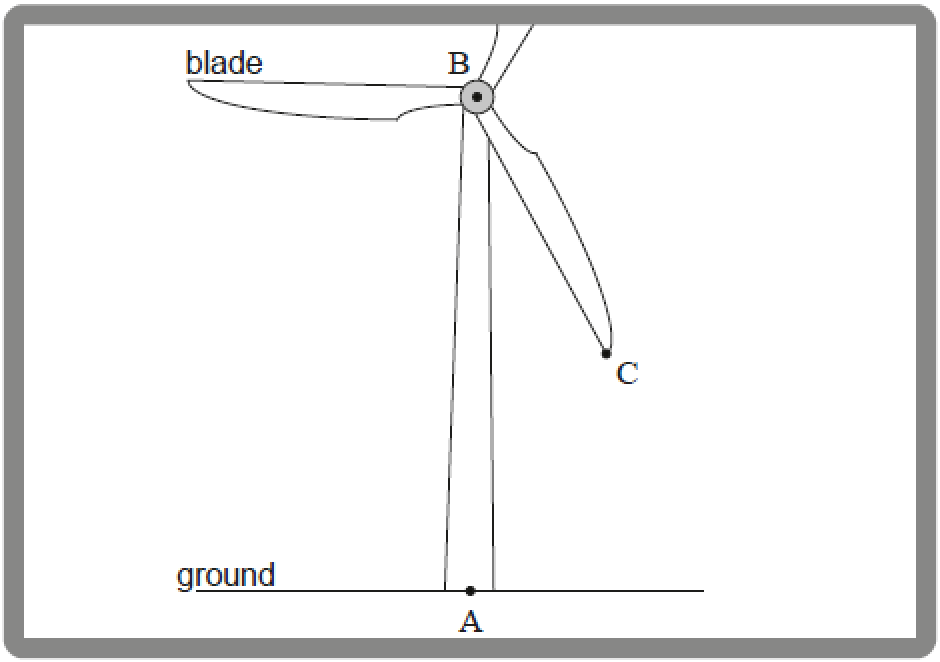
t coordinates $3.55\ (3.54892\ldots)$ **OR** $1.45\ (1.45107\ldots)$ or equivalent ***(A1)***

Note: Award ***A1*** for either t-coordinate seen.

$= 2.10$ seconds $(2.09784\ldots)$ ***A1***

[3 marks]

Looking through his window, Tim has a partial view of the rotating wind turbine. The position of his window means that he cannot see any part of the wind turbine that is **more than $100\ \text{m}$** above the ground. This is illustrated in the following diagram.



(f.i) At any given instant, find the probability that point C is visible from Tim's window. [3]

Markscheme

$5 - 2.09784\ldots$ ***(M1)***

$\frac{(2.902153\ldots)}{5}$ ***(M1)***

$0.580 \ \ (0.580430\ldots)$ ***A1***

[3 marks]

(f.ii) The wind speed increases. The blades rotate at twice the speed, but still at a constant rate.

At any given instant, find the probability that Tim can see point C from his window. Justify your answer. [2]

Markscheme

METHOD 1

changing the frequency/dilation of the graph will not change the proportion of time that point C is visible. ***A1***

$0.580 \ (0.580430\ldots)$ ***A1***

METHOD 2

correct calculation of relevant found values

$\frac{(2.902153\ldots)/2}{5/2}$ ***A1***

$0.580 \ (0.580430\ldots)$ ***A1***

Note: Award ***A0A1*** for an unsupported correct probability.

[2 marks]