Algorithms on graphs [105 marks]

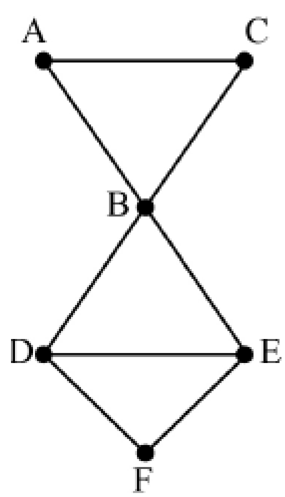
1. [Maximum mark: 17] SPM.2.AHL.TZ0.5

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	B	C	D	E	F
Departure city	A		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

Markscheme



A2

[2 marks]

(b) Write down the adjacency matrix for this graph. [2]

Markscheme

attempt to form an adjacency matrix ***M1***

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A1***

[2 marks]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

Markscheme

raising the matrix to the power six ***(M1)***

50 ***A1***

[2 marks]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

Markscheme

not possible ***A1***

because you must pass through B twice ***R1***

Note: Do not award ***A1R0***.

[2 marks]

The following table shows the least cost to travel between the cities.

		Destination city					
		A	B	C	D	E	F
Departure city	A	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

Markscheme

$a = 230, b = 340$ ***A1A1***

[2 marks]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

Markscheme

A → B → D → E → F → C → A ***(M1)***

90 + 70 + 100 + 210 + 330 + 150 ***(A1)***

(US$) 950 ***A1***

[3 marks]

(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

Markscheme

finding weight of minimum spanning tree ***M1***

$70 + 80 + 100 + 180 =$ (US$) 430 ***A1***

adding in two edges of minimum weight ***M1***

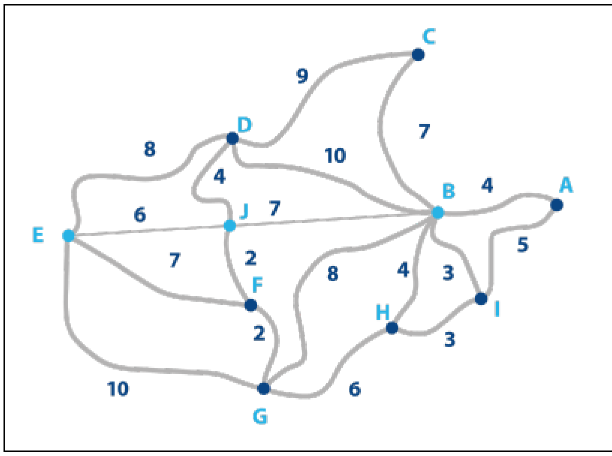
$430 + 90 + 150 =$ (US$) 670 ***A1***

[4 marks]

2. [Maximum mark: 7] EXN.1.AHL.TZ0.11

Nymphenburg Palace in Munich has extensive grounds with 9 points of interest (stations) within them.

These nine points, along with the palace, are shown as the vertices in the graph below. The weights on the edges are the walking times in minutes between each of the stations and the total of all the weights is 105 minutes.



A	Nymphenburg Palace
B	Fountain
C	Magdalenenklause
D	Pagodenburg
E	Large Cascade
F	Pan with Spring
G	Badenburg
H	Amalienburg
I	Summerhouse
J	Bridge

Anders decides he would like to walk along all the paths shown beginning and ending at the Palace (vertex A).

Use the Chinese Postman algorithm, clearly showing all the stages, to find the shortest time to walk along all the paths. [7]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

Odd vertices are B, F, H and I **(M1)A1**

Pairing the vertices **M1**

BF and HI $9 + 3 = 12$

BH and FI $4 + 11 = 15$

BI and FH $3 + 8 = 11$ **A2**

Note: award **A1** for two correct totals.

Shortest time is $105 + 11 = 116$ (minutes) **M1A1**

[7 marks]

3. [Maximum mark: 6] EXM.1.AHL.TZ0.39

Let G be a weighted graph with 6 vertices L, M, N, P, Q, and R. The weight of the edges joining the vertices is given in the table below:

	L	M	N	P	Q	R
L	–	4	3	5	1	4
M	4	–	4	3	2	7
N	3	4	–	2	4	3
P	5	3	2	–	3	4
Q	1	2	4	3	–	5
R	4	7	3	4	5	–

For example the weight of the edge joining the vertices L and N is 3.

(a) Use Prim's algorithm to draw a minimum spanning tree starting at M. [5]

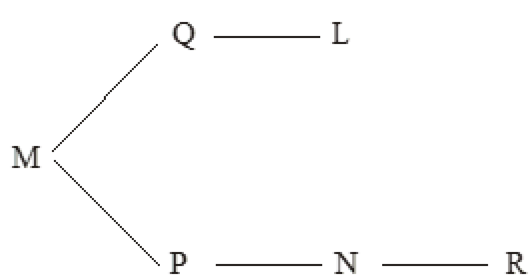
Markscheme

M → Q ***(M1)***

Q → L ***(A1)***

M → P ***(A1)***

P → N → R ***(A1)***



(A1)

Note: There are other correct answers.

[5 marks]

(b) What is the total weight of the tree? [1]

Markscheme

The total weight is $2 + 1 + 3 + 2 + 3 = 11$. ***(A1)***

[1 mark]

4. [Maximum mark: 6] EXM.1.AHL.TZ0.1

The cost adjacency matrix below represents the distance in kilometres, along routes between bus stations.

	A	B	C	D	E
A	-	x	2	6	p
B	x	-	5	7	q
C	2	5	-	3	r
D	6	7	3	-	s
E	p	q	r	s	-

All the values in the matrix are positive, distinct integers.

It is decided to electrify some of the routes, so that it will be possible to travel from any station to any other station solely on electrified routes. In order to achieve this with a minimal total length of electrified routes, Prim's algorithm for a minimal spanning tree is used, starting at vertex A.

The algorithm adds the edges in the following order:

AB AC CD DE.

There is only one minimal spanning tree.

(a) Find with a reason, the value of x. [2]

Markscheme

AB must be the length of the smallest edge from A so $x = 1$. ***R1A1***

[2 marks]

(b) If the total length of the minimal spanning tree is 14, find the value of s. [2]

Markscheme

$1+2+3+s=14 \Rightarrow s=8$ ***M1A1***

[2 marks]

(c) Hence, state, with a reason, what can be deduced about the values of p, q, r. [2]

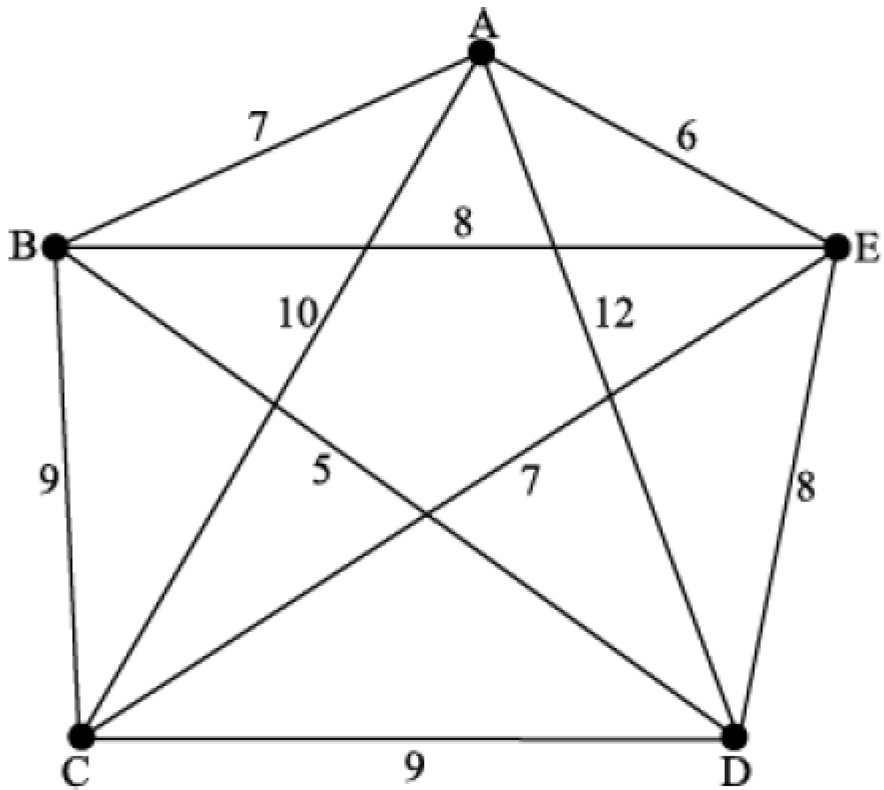
Markscheme

The last minimal edge chosen must connect to E , so since $s=8$ each of p , q, r must be ≥ 9. ***R1A1***

[2 marks]

5. [Maximum mark: 14] EXM.2.AHL.TZ0.19

Let G be the graph below.



(a) Find the total number of Hamiltonian cycles in G, starting at vertex A. Explain your answer. [3]

Markscheme

Starting from vertex A there are 4 choices. From the next vertex there are three choices, etc… ***M1R1***

So the number of Hamiltonian cycles is $4! = 24$. ***A1 N1***

[3 marks]

(b.i) Find a minimum spanning tree for the subgraph obtained by deleting A from G. [3]

Markscheme

Start (for instance) at B, using Prim´s algorithm Then D is the nearest vertex ***M1***

Next E is the nearest vertex ***A1***

Finally C is the nearest vertex So a minimum spanning tree is B → D → E → C ***A1 N1***

[3 marks]

(b.ii) Hence, find a lower bound for the travelling salesman problem for G. [3]

Markscheme

A lower bound for the travelling salesman problem is then obtained by adding the weights of AB and AE to the weight of the minimum ***M1***

spanning tree (ie 20) ***A1***

A lower bound is then 20 + 7 + 6 = 33 ***A1 N1***

[3 marks]

(c) Give an upper bound for the travelling salesman problem for the graph above. [2]

Markscheme

ABCDE is an Hamiltonian cycle ***A1***

Thus an upper bound is given by 7 + 9 + 9 + 8 + 6 = 39 ***A1***

[2 marks]

(d) Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G. [3]

Markscheme

Eliminating C from G a minimum spanning tree is E → A → B → D ***M1***

of weight 18 ***A1***

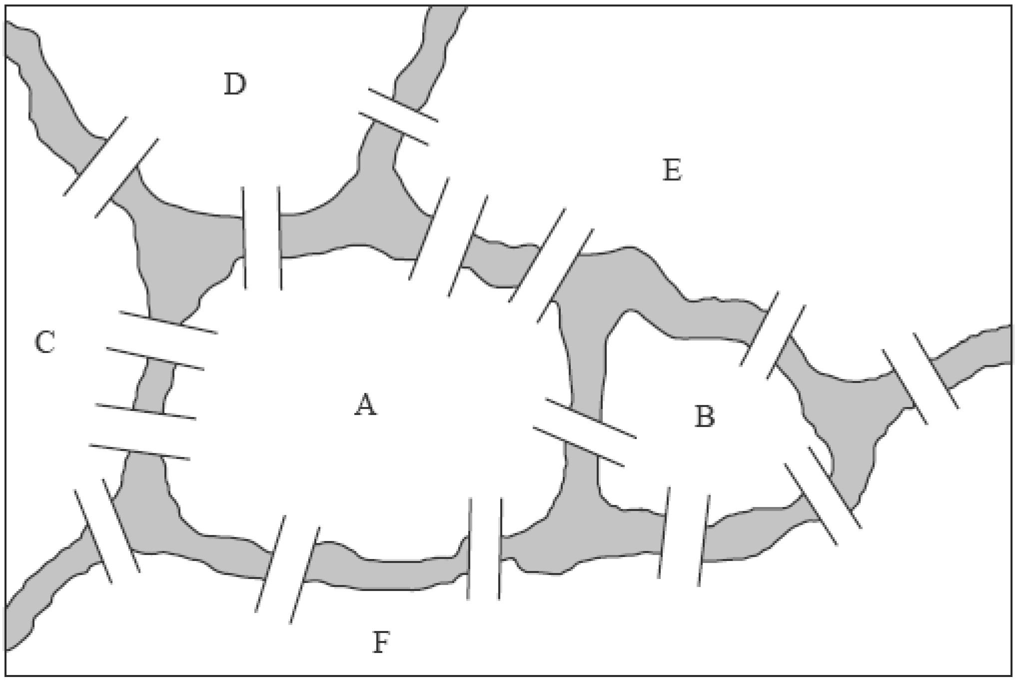
Adding BC to CE$(18 + 9 + 7)$ gives a lower bound of $34 > 33$ ***A1***

So 33 not the best lower bound. ***AG N0***

[3 marks]

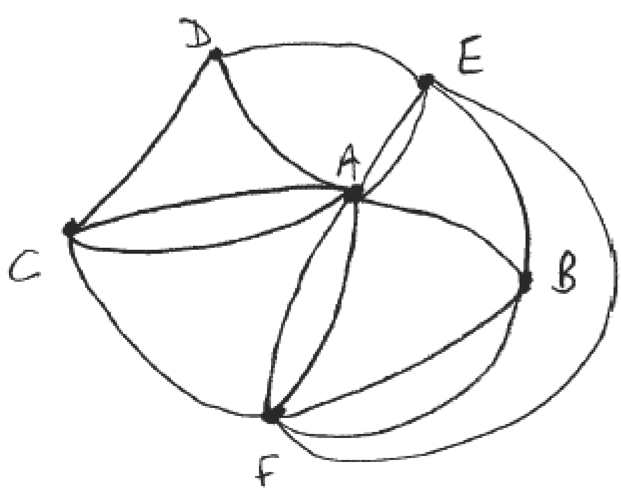
6. [Maximum mark: 12] EXM.2.AHL.TZ0.17

A canal system divides a city into six land masses connected by fifteen bridges, as shown in the diagram below.



(a) Draw a graph to represent this map. [2]

Markscheme



A2

[2 marks]

(b) Write down the adjacency matrix of the graph. [2]

Markscheme

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$ ***A2***

Note: Award A1 for one error or omission, A0 for more than one error or omission. Two symmetrical errors count as one error.

[2 marks]

(c) List the degrees of each of the vertices. [2]

Markscheme

A B C D E F

(8, 4 4, 3 5, 6) ***A2***

Note: Award no more than A1 for one error, A0 for more than one error.

[2 marks]

State with reasons whether or not this graph has

(d.i) an Eulerian circuit. [2]

Markscheme

no, because there are odd vertices ***M1A1***

[2 marks]

(d.ii) an Eulerian trail. [2]

Markscheme

yes, because there are exactly two odd vertices ***M1A1***

[2 marks]

(e) Find the number of walks of length 4 from E to F. [2]

Markscheme

$$M^4 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 309 & 174 & 140 & 118 & 170 & 214 \\ 174 & 117 & 106 & 70 & 122 & 132 \\ 140 & 106 & 117 & 66 & 134 & 138 \\ 118 & 70 & 66 & 53 & 80 & 102 \\ 170 & 122 & 134 & 80 & 157 & 170 \\ 214 & 132 & 138 & 102 & 170 & 213 \end{pmatrix} \end{array}$$ ***(M1)A1***

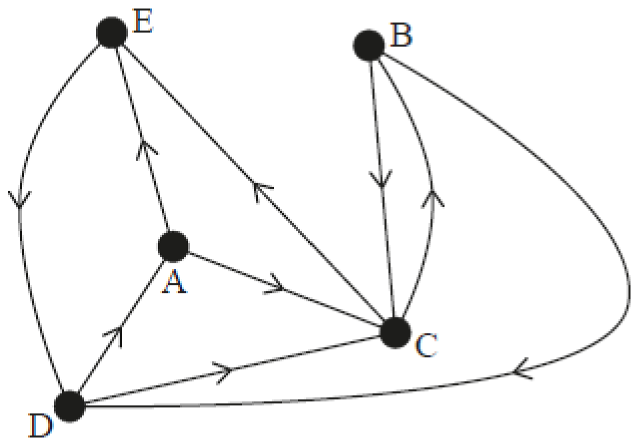
number of walks of length 4 is 170

Note: The complete matrix need not be shown. Only one of the FE has to be shown.

[2 marks]

7. [Maximum mark: 7] 23M.1.AHL.TZ2.8

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.



(a) Construct the adjacency matrix $\boldsymbol{M}$ for this network. [3]

Markscheme

attempt to create a 5x5 adjacency matrix ***(M1)***

$$\boldsymbol{M}=\begin{pmatrix}0&0&1&0&1\\0&0&1&1&0\\0&1&0&0&1\\1&0&1&0&0\\0&0&0&1&0\end{pmatrix}$$

Note: Allow the transposed matrix. Award ***A2*** for all entries correct, ***A1*** if one or two entries are incorrect, ***A0*** otherwise.

Answer presented in markscheme assumes ABCDE ordering of rows and columns; accept other orders provided they are clearly communicated.

Award ***A1*** if the zeroes are replaced by blank cells.

[3 marks]

Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

(b.i) Determine how many possible routes Beatriz could have taken, to travel from E to A. [2]

Markscheme

recognizing need to find $\boldsymbol{M}^7$ ***(M1)***

$$\boldsymbol{M}^7 = \begin{pmatrix} 8 & 8 & 17 & 8 & 13 \\ 8 & 10 & 19 & 17 & 14 \\ 6 & 11 & 16 & 10 & 17 \\ 11 & 8 & 19 & 14 & 10 \\ 2 & 6 & 8 & 11 & 8 \end{pmatrix}$$

2 (routes) ***A1***

[2 marks]

(b.ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [2]

Markscheme

vertices visited in order are

EITHER

$E \to D \to C \to B \to C \to B \to D \to A$ ***A2***

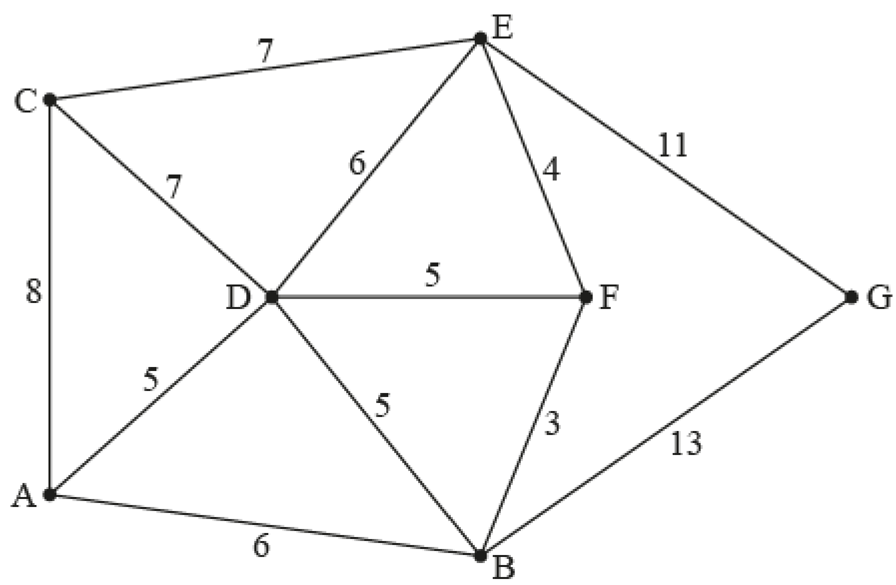
OR

$E \to D \to C \to B \to C \to E \to D \to A$ ***A2***

[2 marks]

8. [Maximum mark: 17] 23M.2.AHL.TZ1.4

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

Markscheme

there are more than two vertices with odd degree ***R1***

so it is not possible to travel along each road exactly once ***A1***

Note: Do not award ***R0A1***.

Award ***R1*** for "There are 4 vertices with odd degree".

[2 marks]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	a	b
D	5	5	7		6	5	c
E	11	7	7	6		4	11
F	9	3	a	5	4		d
G	19	13	b	c	11	d	

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

Markscheme

$a = 11, b = 18, c = 17, d = 15$ ***A2***

Note: Award ***A1*** for any one correct, ***A2*** for all four correct.

[2 marks]

(c) Use the nearest neighbour algorithm, starting at vertex G to find an upper bound for the travelling salesman problem. [3]

Markscheme

attempt to use nearest neighbour algorithm *(M1)*

Note: Award ***M1*** for first 3 vertices correct or $11, 4, 3$ seen.

$\mathrm{G–E–F–B–D–A–C(–E)–G}$ **OR** $11 + 4 + 3 + 5 + 5 + 8 +$ their b *(A1)*

upper bound = $54\,(\mathrm{km})$ ***A1***

[3 marks]

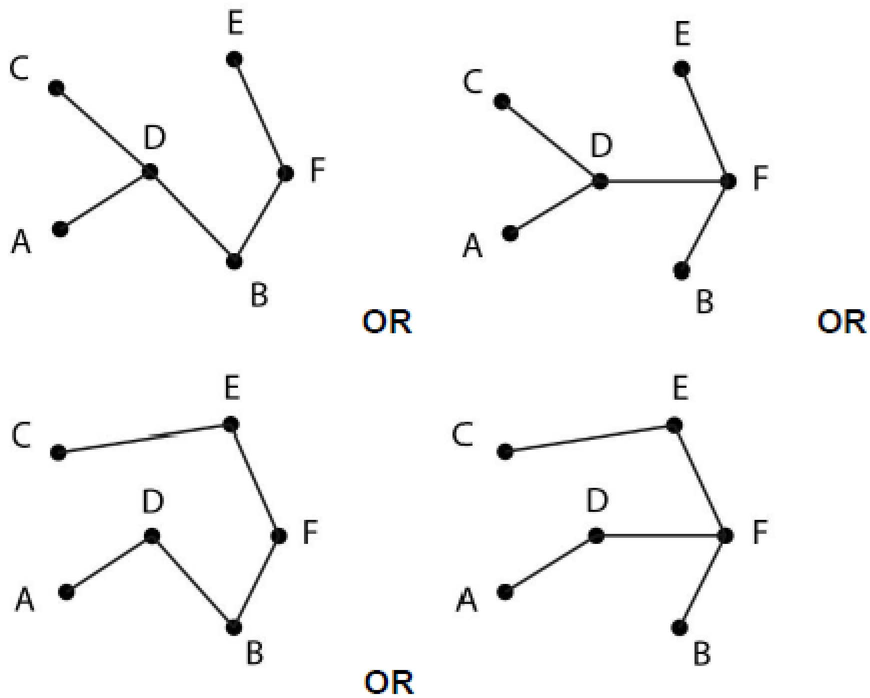
(d.i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A,\ B,\ C,\ D,\ E,\ F.}$ [2]

Markscheme

a diagram of **any** spanning tree of the subgraph ABCDEF *(A1)*

attempt at Kruskal's algorithm or Prim's algorithm *(M1)*

e.g. edges BF (3), EF (4) and an edge of length 5 listed or seen in any spanning tree



A1

(d.ii) Write down the total weight of the minimum spanning tree. [2]

Markscheme

$24\,(\mathrm{km})$ ***A1***

Note: ***FT*** from their sketch, only if it is a spanning tree. It is not required to see the edge lengths on the sketch, since they are given in the question.

[2 marks]

(e) Hence find a lower bound for the travelling salesman problem. [2]

Markscheme

adding vertex G's two shortest edges to their part (d)(ii) ***(M1)***

$$24 + 11 + 13$$

$$= 48$$ ***A1***

[2 marks]

(f) Explain one way in which an improved lower bound could be found. [1]

Markscheme

try removing a different vertex ***A1***

[1 mark]

It is found that the optimum solution starting at A is actually A–C–E–G–B–F–D–A.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

Markscheme

recognize 7 edges in optimum route ***(M1)***

Note: Award ***M1*** for a total length of 52 seen.

subtracting $0.5\times$ edges from 52 ***(M1)***

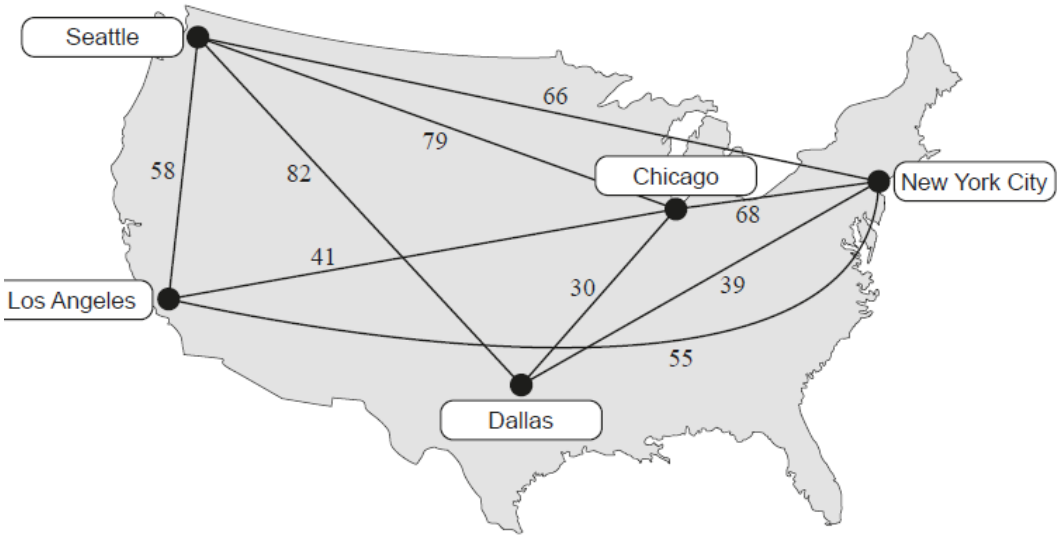
$$52 - 7 \times 0.5$$

$= 48.5\,(\mathrm{km})$ ***A1***

[3 marks]

9. [Maximum mark: 19] 23M.2.AHL.TZ2.4

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($\$$) between cities.



(a) Explain why the graph can be described as “connected”, but not “complete”. [2]

Markscheme

any city can be travelled to or from any other city (so is connected) ***R1***

EITHER

but there is no direct flight between Los Angeles and Dallas (for example) ***R1***

OR

but not every vertex has degree 4 ***R1***

Note: Accept equivalent statements for the cities being connected and the graph not being complete.

[2 marks]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

Markscheme

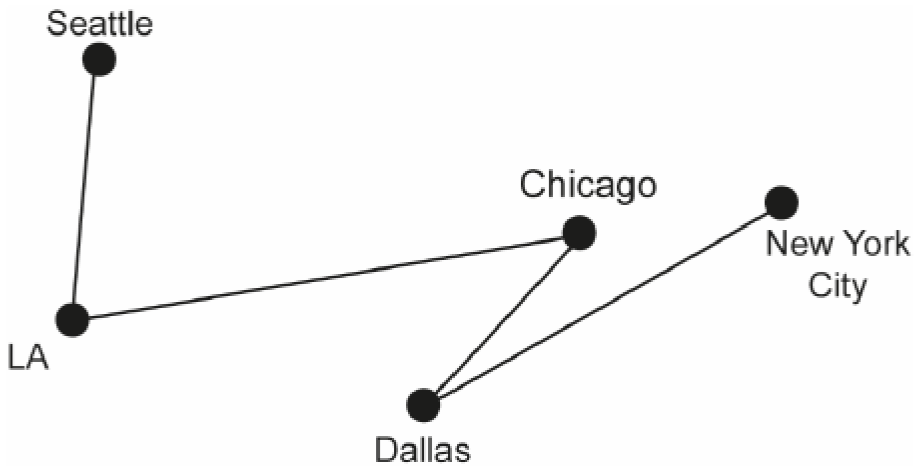
edge CD selected first ***M1***

DN,

CL,

LS ***A1***

Note: Award marks if the answers are written as sums in the correct order. ***M1*** if 30 is seen first, ***A1*** for $30 + 39 + 41 + 58$.



A1

Note: The final ***A1*** can be awarded independently. Award ***M0A0A1*** for a correct MST graph with no other working. Award ***M1A0A1*** if Prim's algorithm is seen to be used correctly with CD first.

[3 marks]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Markscheme

$2 \times \mathrm{MST}$ weight ***(M1)***

$= \$336$ ***A1***

Note: Allow any integer multiple (> 1) of MST weight for ***M1,*** and if correctly calculated, award ***M1A1.***

[2 marks]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

Markscheme

attempt at nearest neighbour algorithm ***M1***

order is $\mathrm{LA} \rightarrow \mathrm{D} \rightarrow \mathrm{C} \rightarrow \mathrm{NYC} \rightarrow \mathrm{S} \rightarrow \mathrm{LA}$ ***A1***

Note: Award ***M1*** for a route that begins with LA and then D, this includes seeing 26 as the first value in a sum. Award ***A1*** if $26+30+68+66+58$ seen in order.

Note: Award ***M1A0*** for an incorrect first nearest neighbour proceeding 'correctly' to the next vertex. For example, LA to C and then C to D.

upper bound is $(26+30+68+66+58=)\ \$248$ ***A1***

Note: Award ***M1A0*** for correct nearest neighbour algorithm starting from a vertex other than LA. Condone the correct tour written backwards i.e. $58+66+68+30+26=248$

[3 marks]

(e.i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem. [3]

Markscheme

attempt to find MST of L, N, D and S ***(M1)***

by deleting C, Kruskal gives MST for the remainder as LD, DN, LS weight 123 ***(A1)***

(lower bound is therefore $123 + (30 + 41) =)\$194$ ***A1***

Note: Award ***(M1)*** for a graph or list of edges that does not include C.

Award ***(A1)*** if $26 + 39 + 58$ seen in any order.

[3 marks]

(e.ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [2]

Markscheme

by deleting S, Kruskal gives MST for the remainder as LD, DC, DN weight 95 ***(A1)***

(lower bound is therefore $95 + (58 + 66) =)\ \$219$ ***A1***

Note: Award *(A1)* if $26 + 30 + 39$ seen in any order.

[2 marks]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

Markscheme

$219 \leq C \leq 248$ ***A1A1***

Note: Award ***A1*** for $219 \leq C$ and ***A1*** for $C \leq 248$. Award at most ***A1A0*** for $219 < C < 248$. ***FT*** for their values from part (e) if higher value from (e)(i) and (e)(ii) used for the lower bound, and part (d) for the upper.

[2 marks]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

Markscheme

any valid tour, within their interval from part (f), from any starting point **OR** any valid tour that starts and finishes at N ***(M1)***

valid tour starting point N **AND** within their interval ***A1***
e.g. NDCLSN (weight 234)

Note: If part (f) not correct, only award ***A1FT*** if their valid tour begins and ends at N **AND** lies within **BOTH** their interval (including if one-sided) in part (f) **AND** $219 \leq C \leq 248$.

If no response in the form of an interval seen in part (f) then award ***M1A0*** for a valid tour beginning and ending at N **AND** within $219 \leq C \leq 248$.

[2 marks]