Algorithms on graphs [105 marks]

1. [Maximum mark: 17] SPM.2.AHL.TZ0.5

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

(b) Write down the adjacency matrix for this graph. [2]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

The following table shows the least cost to travel between the cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

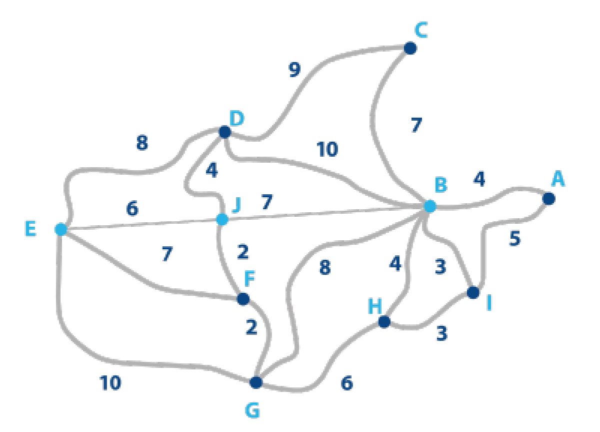
(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

2. [Maximum mark: 7] EXN.1.AHL.TZ0.11

Nymphenburg Palace in Munich has extensive grounds with 9 points of interest (stations) within them.

These nine points, along with the palace, are shown as the vertices in the graph below. The weights on the edges are the walking times in minutes between each of the stations and the total of all the weights is 105 minutes.



A	Nymphenburg Palace
B	Fountain
C	Magdalenenklause
D	Pagodenburg
E	Large Cascade
F	Pan with Spring
G	Badenburg
H	Amalienburg
I	Summerhouse
J	Bridge

Anders decides he would like to walk along all the paths shown beginning and ending at the Palace (vertex A).

Use the Chinese Postman algorithm, clearly showing all the stages, to find the shortest time to walk along all the paths. [7]

3. [Maximum mark: 6] EXM.1.AHL.TZ0.39

Let G be a weighted graph with 6 vertices L, M, N, P, Q, and R. The weight of the edges joining the vertices is given in the table below:

	L	M	N	P	Q	R
L	–	4	3	5	1	4
M	4	–	4	3	2	7
N	3	4	–	2	4	3
P	5	3	2	–	3	4
Q	1	2	4	3	–	5
R	4	7	3	4	5	–

For example the weight of the edge joining the vertices L and N is 3.

(a) Use Prim's algorithm to draw a minimum spanning tree starting at M. [5]

(b) What is the total weight of the tree? [1]

4. [Maximum mark: 6] EXM.1.AHL.TZ0.1

The cost adjacency matrix below represents the distance in kilometres, along routes between bus stations.

	A	B	C	D	E
A	-	x	2	6	p
B	x	-	5	7	q
C	2	5	-	3	r
D	6	7	3	-	s
E	p	q	r	s	-

All the values in the matrix are positive, distinct integers.

It is decided to electrify some of the routes, so that it will be possible to travel from any station to any other station solely on electrified routes. In order to achieve this with a minimal total length of electrified routes, Prim's algorithm for a minimal spanning tree is used, starting at vertex A.

The algorithm adds the edges in the following order:

AB AC CD DE.

There is only one minimal spanning tree.

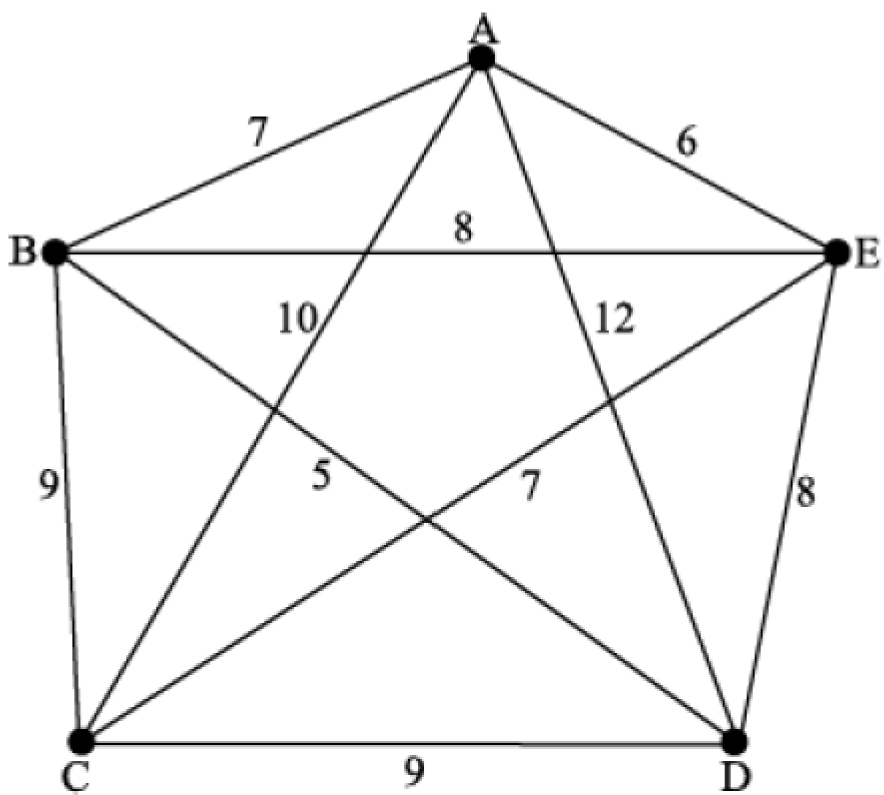
(a) Find with a reason, the value of x. [2]

(b) If the total length of the minimal spanning tree is 14, find the value of s. [2]

(c) Hence, state, with a reason, what can be deduced about the values of p, q, r. [2]

5. [Maximum mark: 14] EXM.2.AHL.TZ0.19

Let G be the graph below.



(a) Find the total number of Hamiltonian cycles in G, starting at vertex A. Explain your answer. [3]

(b.i) Find a minimum spanning tree for the subgraph obtained by deleting A from G. [3]

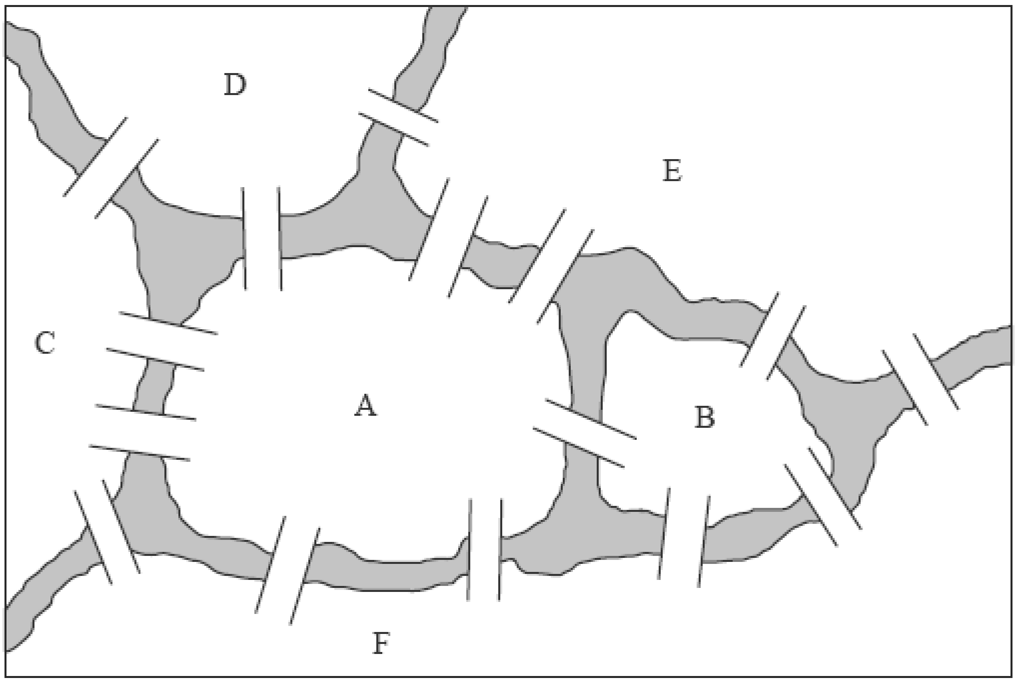
(b.ii) Hence, find a lower bound for the travelling salesman problem for G. [3]

(c) Give an upper bound for the travelling salesman problem for the graph above. [2]

(d) Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G. [3]

6. [Maximum mark: 12] EXM.2.AHL.TZ0.17

A canal system divides a city into six land masses connected by fifteen bridges, as shown in the diagram below.



(a) Draw a graph to represent this map. [2]

(b) Write down the adjacency matrix of the graph. [2]

(c) List the degrees of each of the vertices. [2]

State with reasons whether or not this graph has

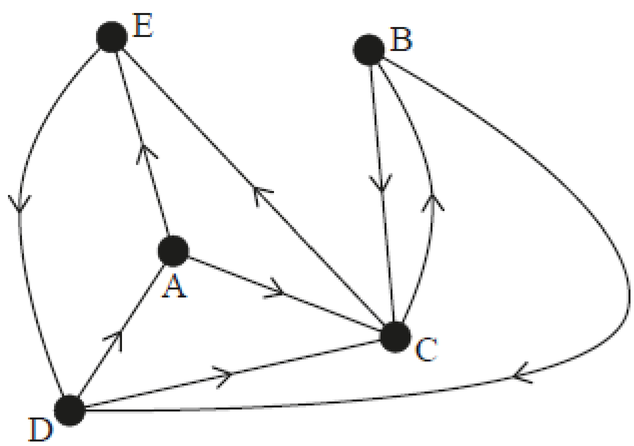
(d.i) an Eulerian circuit. [2]

(d.ii) an Eulerian trail. [2]

(e) Find the number of walks of length 4 from E to F. [2]

7. [Maximum mark: 7] 23M.1.AHL.TZ2.8

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.



(a) Construct the adjacency matrix $\boldsymbol{M}$ for this network. [3]

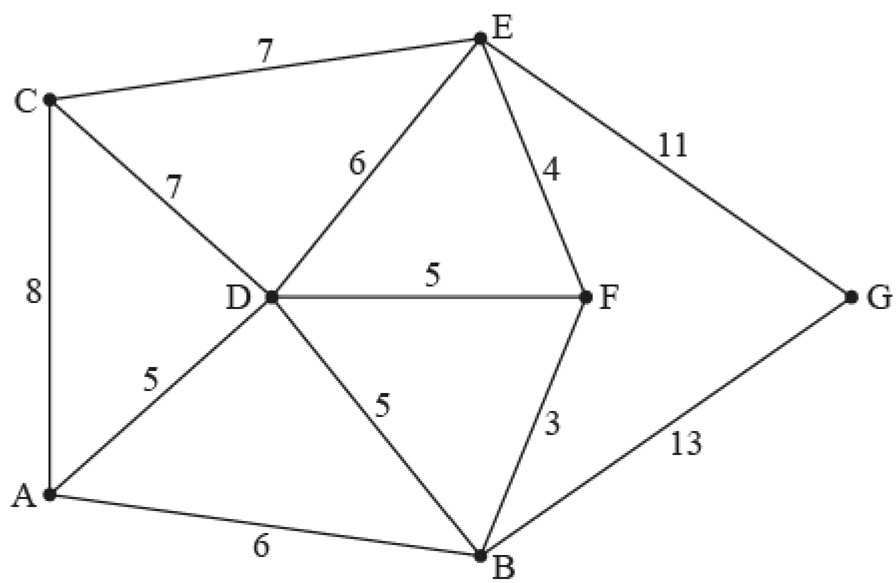
Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

(b.i) Determine how many possible routes Beatriz could have taken, to travel from E to A. [2]

(b.ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [2]

8. [Maximum mark: 17] 23M.2.AHL.TZ1.4

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	a	b
D	5	5	7		6	5	c
E	11	7	7	6		4	11
F	9	3	a	5	4		d
G	19	13	b	c	11	d	

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

(c) Use the nearest neighbour algorithm, starting at vertex G to find an upper bound for the travelling salesman problem. [3]

(d.i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F}$. [2]

(d.ii) Write down the total weight of the minimum spanning tree. [2]

(e) Hence find a lower bound for the travelling salesman problem. [2]

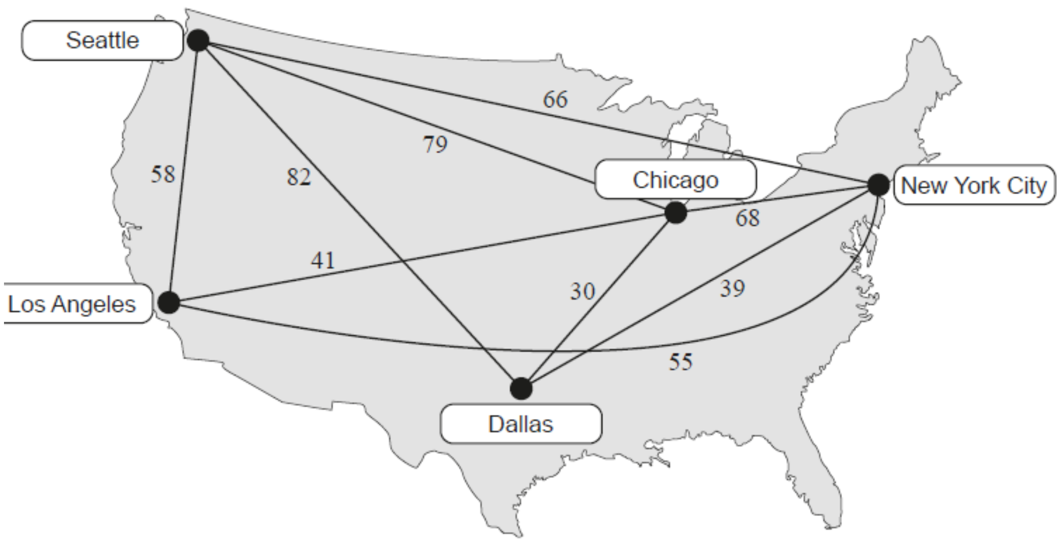
(f) Explain one way in which an improved lower bound could be found. [1]

It is found that the optimum solution starting at A is actually $\mathrm{A–C–E–G–B–F–D–A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

9. [Maximum mark: 19] 23M.2.AHL.TZ2.4

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($\$$) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

(e.i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem. [3]

(e.ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [2]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]