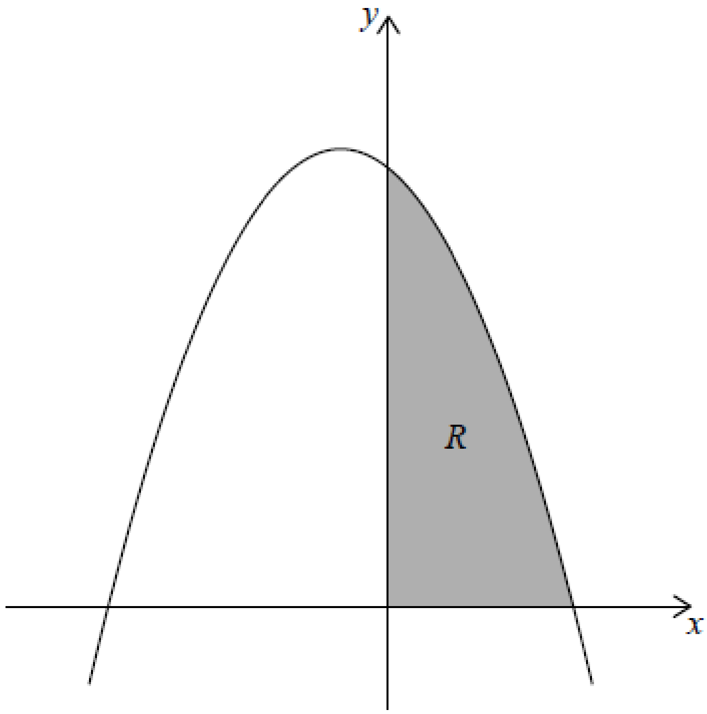
Application of integration [53 marks]

1. [Maximum mark: 5] SPM.1.SL.TZ0.10

The following diagram shows part of the graph of $f(x) = (6 - 3x)(4 + x), x \in \mathbb{R}$. The shaded region *R* is bounded by the x-axis, y-axis and the graph of f.



(a) Write down an integral for the area of region *R*. [2]

Markscheme

$A = \int_0^2 (6 - 3x)(4 + x)\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for the limits $x = 0, x = 2$. Award ***A1*** for an integral of $f(x)$.

[2 marks]

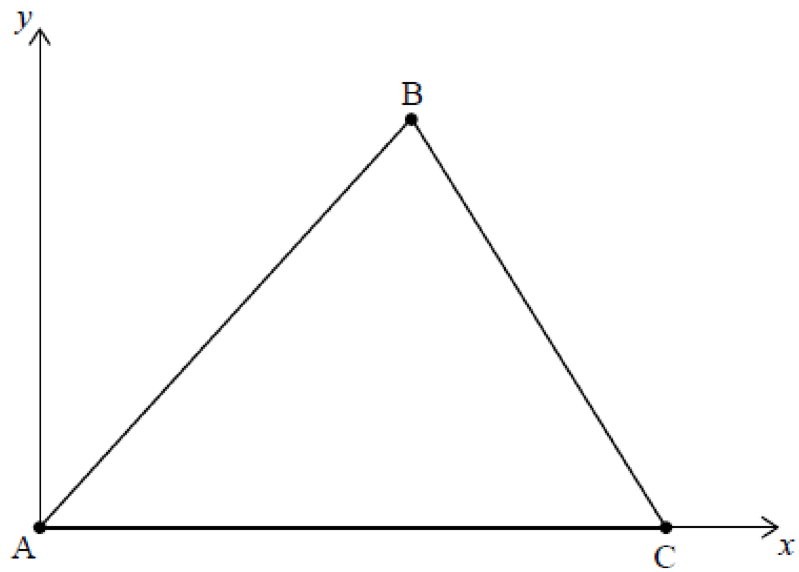
(b) Find the area of region *R*. [1]

Markscheme

28 ***A1***

[1 mark]

(c) The three points A(0, 0) , B(3, 10) and C(a, 0) define the vertices of a triangle.



Find the value of a, the x-coordinate of C, such that the area of the triangle is equal to the area of region *R*. [2]

Markscheme

$28 = 0.5 \times a \times 10$ ***M1***

$5.6 \left(\frac{28}{5}\right)$ ***A1***

[2 marks]

2. [Maximum mark: 12] EXN.2.SL.TZ0.3

The rate of change of the height (h) of a ball above horizontal ground, measured in metres, t seconds after it has been thrown and until it hits the ground, can be modelled by the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 11.4 - 9.8t$$

The height of the ball when $t = 0$ is $1.2\,\mathrm{m}$.

(a) Find an expression for the height h of the ball at time t. [6]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$h = \int (11.4 - 9.8t)\mathrm{d}t$ **M1**

$h = 11.4t - 4.9t^2\,(+c)$ **A1A1**

When $t = 0,\ h = 1.2$ **(M1)**

$c = 1.2$ **(A1)**

$(h =)1.2 + 11.4t - 4.9t^2$ **A1**

[6 marks]

(b.i) Find the value of t at which the ball hits the ground. [2]

Markscheme

$2.43\ (2.42741\ldots)$ seconds **(M1)A1**

[2 marks]

(b.ii) Hence write down the domain of h. [1]

Markscheme

$0 \leq t \leq 2.43$ **A1**

Note: Accept $0 \leq t < 2.43$.

[1 mark]

(c) Find the range of h. [3]

Markscheme

Maximum value is $7.83061\ldots$ **(M1)**

Range is $0 \leq h \leq 7.83$ **A1A1**

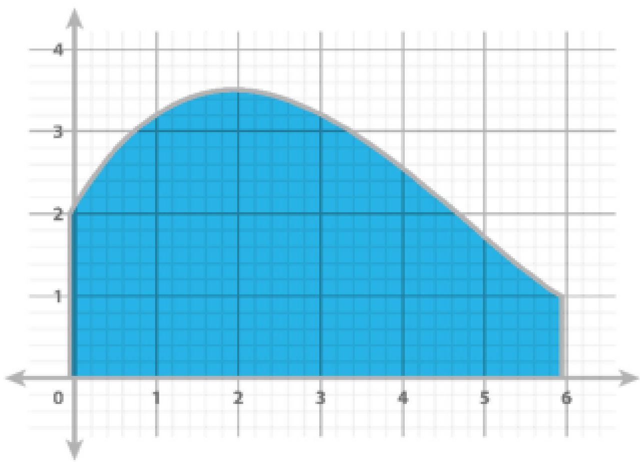
Note: Accept $0 \leq h < 7.83$.

[3 marks]

3. [Maximum mark: 16] EXN.2.SL.TZ0.6

A theatre set designer is designing a piece of flat scenery in the shape of a hill. The scenery is formed by a curve between two vertical edges of unequal height. One edge is 2 metres high and the other is 1 metre high. The width of the scenery is 6 metres.

A coordinate system is formed with the origin at the foot of the 2 metres high edge. In this coordinate system the highest point of the cross-section is at $(2,\ 3.5)$.



A set designer wishes to work out an approximate value for the area of the scenery $(A\,\mathrm{m}^2)$.

(a) Explain why $A < 21$. [1]

Markscheme

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The area A is less than the rectangle containing the cross-section which is equal to $6 \times 3.5 = 21$ **R1**

Note: $6 \times 3.5 = 21$ is not sufficient for **R1**.

[1 mark]

(b) By dividing the area between the curve and the x-axis into two trapezoids of unequal width show that $A > 14.5$, justifying the direction of the inequality. [4]

Markscheme

$\frac{1}{2} \times 2 \times (2 + 3.5) + \frac{1}{2} \times 4 \times (3.5 + 1)$ **(M1)(A1)**

$= 14.5$ **A1**

This is an underestimate as the trapezoids are enclosed by (are under) the curve. **R1**

Note: This can be shown in a diagram.

[4 marks]

In order to obtain a more accurate measure for the area the designer decides to model the curved edge with the polynomial $h(x) = ax^3 + bx^2 + cx + d \;\; a, b, c, d \in \mathbb{R}$ where h metres is the height of the curved edge a horizontal distance $x\,\mathrm{m}$ from the origin.

(c) Write down the value of d. [1]

Markscheme

$h(0) = 2 \Rightarrow d = 2$ **A1**

[1 mark]

(d) Use differentiation to show that $12a + 4b + c = 0$. [2]

Markscheme

$h\prime(x) = 3ax^2 + 2bx + c$ **A1**

$h\prime(2) = 0$ **M1**

hence $12a + 4b + c = 0$ **AG**

[2 marks]

(e) Determine two other linear equations in a, b and c. [3]

Markscheme

Substitute the points $(2,\ 3.5)$ and $(6,\ 1)$ **(M1)**

$8a + 4b + 2c + 2 = 3.5\ (8a + 4b + 2c = 1.5)$

and

$216a + 36b + 6c + 2 = 1\ (216a + 36b + 6c = -1)$ **A1A1**

[3 marks]

(f) Hence find an expression for $h(x)$.

[3]

Markscheme

Solve on a GDC **(M1)**

$h(x) = 0.0365x^3 - 0.521x^2 + 1.65x + 2$ **A2**

$\left(h(x) = 0.0364583\ldots x^3 - 0.520833\ldots x^2 + 1.64583\ldots x + 2\right)$

[3 marks]

(g) Use the expression found in (f) to calculate a value for A. [2]

Markscheme

$\int_0^6 0.0364583\ldots x^3 - 0.520833\ldots x^2 + 1.64583\ldots x + 2\,\mathrm{d}x$

$= 15.9\ (15.9374\ldots)\ \left(\mathrm{m}^2\right)$ **(M1)A1**

Note: Accept $16.0\ (16.014)$ from the three significant figure answer to part (g).

[2 marks]

4. [Maximum mark: 7] 21M.1.SL.TZ2.13

A company produces and sells electric cars. The company's profit, P, in thousands of dollars, changes based on the number of cars, x, they produce per month.

The rate of change of their profit from producing x electric cars is modelled by

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -1.6x + 48,\ x \geq 0.$$

The company makes a profit of 260 (thousand dollars) when they produce 15 electric cars.

(a) Find an expression for P in terms of x. [5]

Markscheme

recognition of need to integrate (*eg* reverse power rule or integral symbol) ***(M1)***

$P(x) = -0.8x^2 + 48x\,(+c)$ ***A1A1***

$260 = -0.8 \times (15)^2 + 48 \times (15) + c$ ***(M1)***

Note: Award ***M1*** for correct substitution of $x = 15$ and $P = 260$. A constant of integration must be seen (can be implied by a correct answer).

$c = -280$

$P(x) = -0.8x^2 + 48x - 280$ ***A1***

[5 marks]

(b) The company regularly increases the number of cars it produces.

Describe how their profit changes if they increase production to over 30 cars per month and up to 50 cars per month. Justify your answer. [2]

Markscheme

profit will decrease (with each new car produced) ***A1***

EITHER

because the profit function is decreasing / the gradient is negative / the rate of change of P is negative ***R1***

OR

$\int_{30}^{50} -1.6x + 48 (\mathrm{d}\ x) = -320$ ***R1***

OR

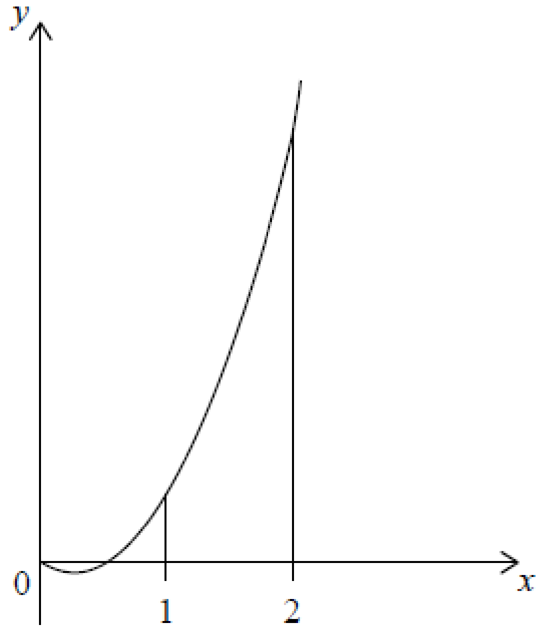
evidence of finding $P(30) = 440$ and $P(50) = 120$ ***R1***

Note: Award at most ***R1A0*** if $P(30)$ or $P(50)$ or both have incorrect values.

[2 marks]

5. [Maximum mark: 6] 18M.1.SL.TZ2.S_2

Let $f\left(x\right) = 6x^2 - 3x$. The graph of f is shown in the following diagram.



(a) Find $\int \left(6x^2 - 3x\right)\mathrm{d}x$. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2x^3 - \frac{3x^2}{2} + c \quad \left(\text{accept } \frac{6x^3}{3} - \frac{3x^2}{2} + c\right)$ ***A1A1 N2***

Notes: Award ***A1A0*** for both correct terms if +*c* is omitted.
Award ***A1A0*** for one correct term *eg* $2x^3 + c$.
Award ***A1A0*** if both terms are correct, but candidate attempts further working to solve for *c*.

[2 marks]

(b) Find the area of the region enclosed by the graph of f, the x-axis and the lines $x = 1$ and $x = 2$.

[4]

Markscheme

substitution of limits or function ***(A1)***

eg $\int_1^2 f(x)\,\mathrm{d}x,\ \left[2x^3 - \frac{3x^2}{2}\right]_1^2$

substituting limits into their integrated function and subtracting ***(M1)***

eg $\frac{6\times 2^3}{3} - \frac{3\times 2^2}{2} - \left(\frac{6\times 1^3}{3} + \frac{3\times 1^2}{2}\right)$

Note: Award ***M0*** if substituted into original function.

correct working ***(A1)***

eg $\frac{6\times 8}{3} - \frac{3\times 4}{2} - \frac{6\times 1}{3} + \frac{3\times 1}{2},\ (16-6) - \left(2 - \frac{3}{2}\right)$

$\frac{19}{2}$ ***A1 N3***

[4 marks]

6. [Maximum mark: 7] 18M.2.SL.TZ1.S_4

Let $g(x) = -(x - 1)^2 + 5$.

(a) Write down the coordinates of the vertex of the graph of g. [1]

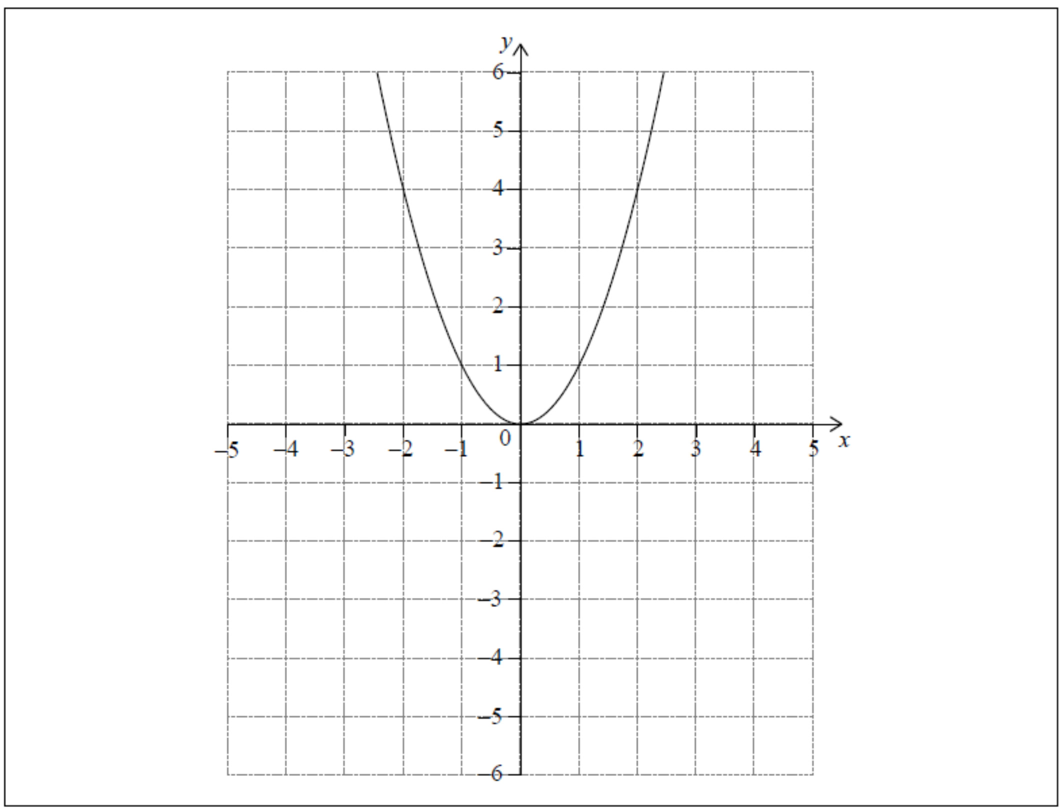
Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

(1,5) (exact) ***A1 N1***

[1 mark]

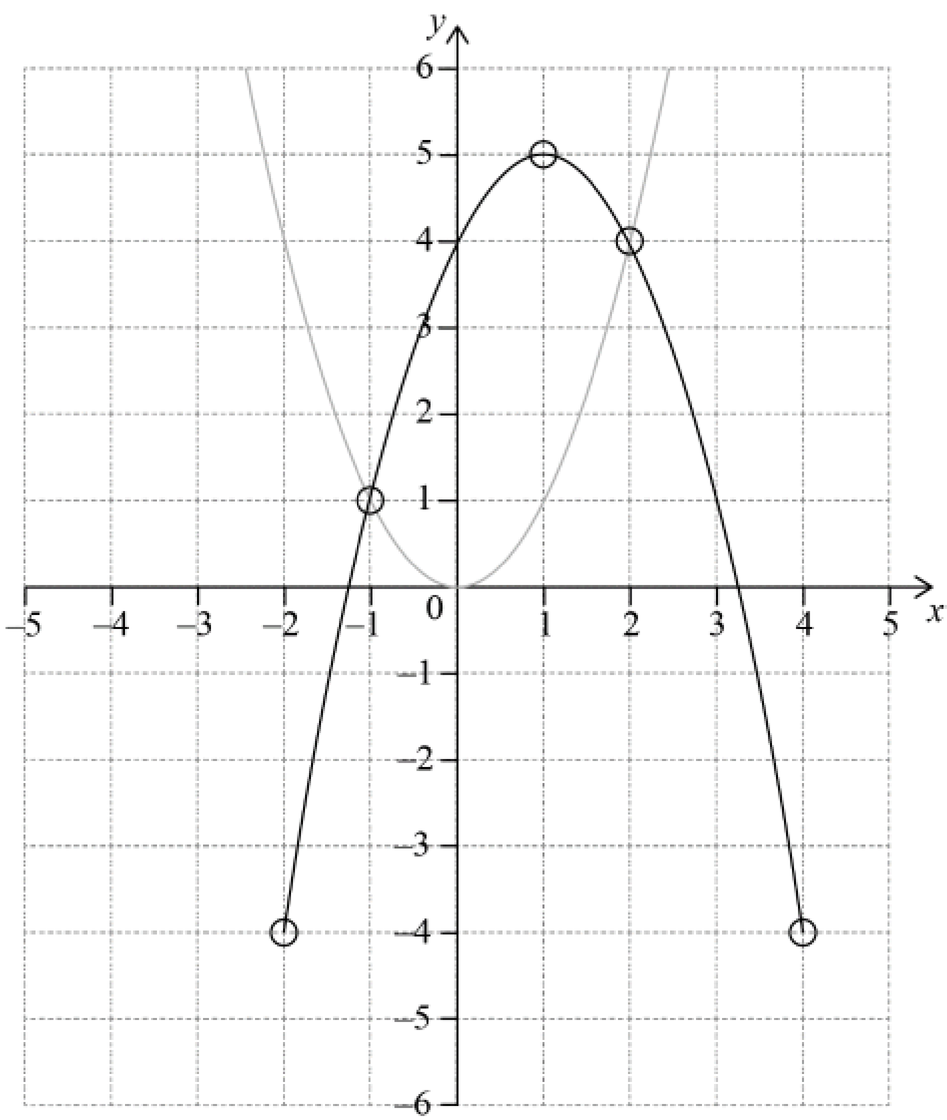
Let $f(x) = x^2$. The following diagram shows part of the graph of f.



The graph of g intersects the graph of f at $x = -1$ and $x = 2$.

(b) On the grid above, sketch the graph of g for $-2 \le x \le 4$. [3]

Markscheme



A1A1A1 N3

Note: The shape must be a concave-down parabola.

Only if the shape is correct, award the following for points in circles:

A1 for vertex,

A1 for correct intersection points,

A1 for correct endpoints.

[3 marks]

(c) Find the area of the region enclosed by the graphs of f and g. [3]

Markscheme

integrating and subtracting functions (in any order) ***(M1)***

eg $\int f - g$

correct substitution of limits or functions (accept missing dx, but do not accept any errors, including extra bits) ***(A1)***

eg $\int_{-1}^{2} g - f, \ \int -(x-1)^2 + 5 - x^2$

area = 9 (exact) ***A1 N2***

[3 marks]