Application of integration [53 marks]
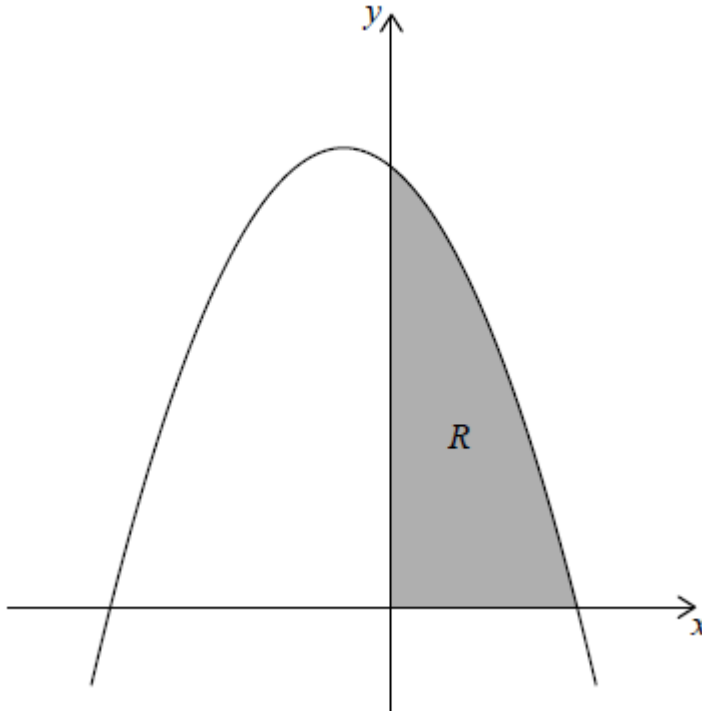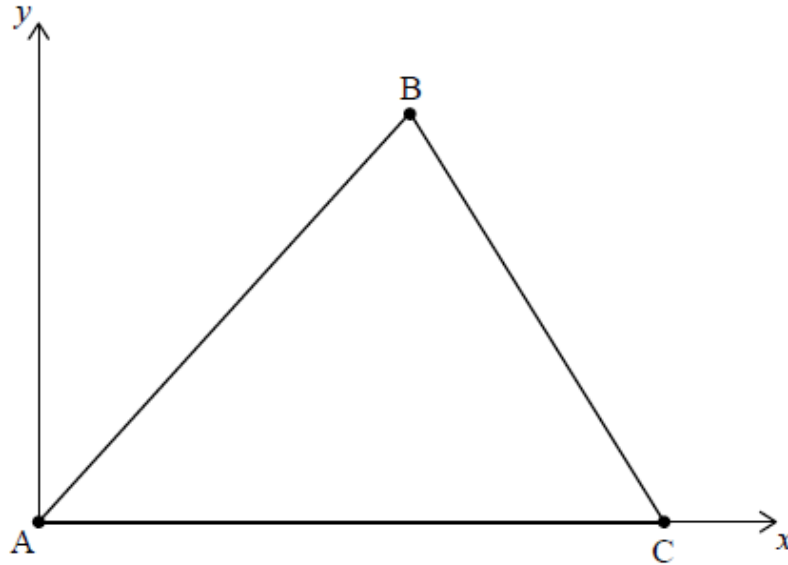
1. [Maximum mark: 5]

The following diagram shows part of the graph of
$f(x) = (6 - 3x)(4 + x), x \in \mathbb{R}.$ The shaded region R is bounded by the
x-axis, y-axis and the graph of f.

(a) Write down an integral for the area of region R. [2]

(b) Find the area of region R. [1]

(c) The three points A$(0, 0)$, B$(3, 10)$ and C$(a, 0)$ define the vertices
of a triangle.

Find the value of a, the x-coordinate of C, such that the area of the triangle is equal to the area of region R.

2.   [Maximum mark: 12]   EXN.2.SL.TZ0.3

The rate of change of the height (h) of a ball above horizontal ground, measured in metres, t seconds after it has been thrown and until it hits the ground, can be modelled by the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 11.4 - 9.8t$$

The height of the ball when $t = 0$ is $1.2\,\mathrm{m}$.

(a)   Find an expression for the height h of the ball at time t.   [6]

(b.i)   Find the value of t at which the ball hits the ground.   [2]

(b.ii)   Hence write down the domain of h.   [1]

(c)   Find the range of h.   [3]

3. [Maximum mark: 16]

A theatre set designer is designing a piece of flat scenery in the shape of a hill. The scenery is formed by a curve between two vertical edges of unequal height. One edge is 2 metres high and the other is 1 metre high. The width of the scenery is 6 metres.

A coordinate system is formed with the origin at the foot of the 2 metres high edge. In this coordinate system the highest point of the cross-section is at $(2, \ 3.5)$.
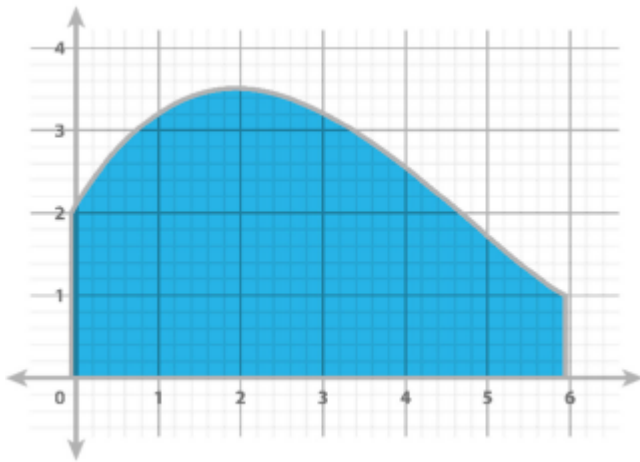
A set designer wishes to work out an approximate value for the area of the scenery $\left(A\,\mathrm{m}^2\right)$.

(a) Explain why $A < 21$. [1]

(b) By dividing the area between the curve and the x-axis into two trapezoids of unequal width show that $A > 14.5$, justifying the direction of the inequality. [4]

In order to obtain a more accurate measure for the area the designer decides to model the curved edge with the polynomial $h(x) = ax^3 + bx^2 + cx + d \ \ a, \, b, \, c, \, d \in \mathbb{R}$ where h metres is the height of the curved edge a horizontal distance $x\,\mathrm{m}$ from the origin.

(c) Write down the value of d. [1]

(d)   Use differentiation to show that $12a + 4b + c = 0$.   [2]

(e)   Determine two other linear equations in a, b and c.   [3]

(f)   Hence find an expression for $h(x)$.   [3]

(g)   Use the expression found in (f) to calculate a value for A.   [2]

4.   [Maximum mark: 7]

A company produces and sells electric cars. The company's profit, P, in thousands of dollars, changes based on the number of cars, x, they produce per month.

The rate of change of their profit from producing x electric cars is modelled by

$\frac{\mathrm{d}P}{\mathrm{d}x} = -1.6x + 48, \ x \geq 0.$

The company makes a profit of 260 (thousand dollars) when they produce 15 electric cars.
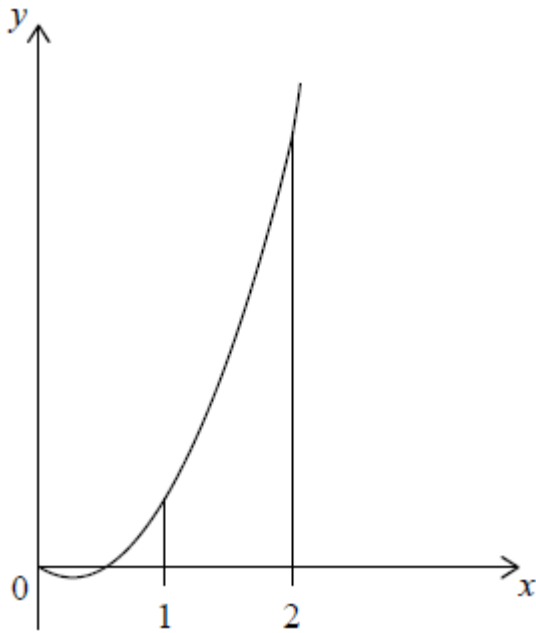
(a)   Find an expression for P in terms of x.   [5]

(b)   The company regularly increases the number of cars it produces.

        Describe how their profit changes if they increase production to over 30 cars per month and up to 50 cars per month. Justify your answer.   [2]

5. [Maximum mark: 6]

Let $f(x) = 6x^2 - 3x$. The graph of f is shown in the following diagram.

(a) Find $\int \left(6x^2 - 3x\right) dx$. [2]

(b) Find the area of the region enclosed by the graph of f, the x-axis and the lines $x = 1$ and $x = 2$. [4]

6.   [Maximum mark: 7]
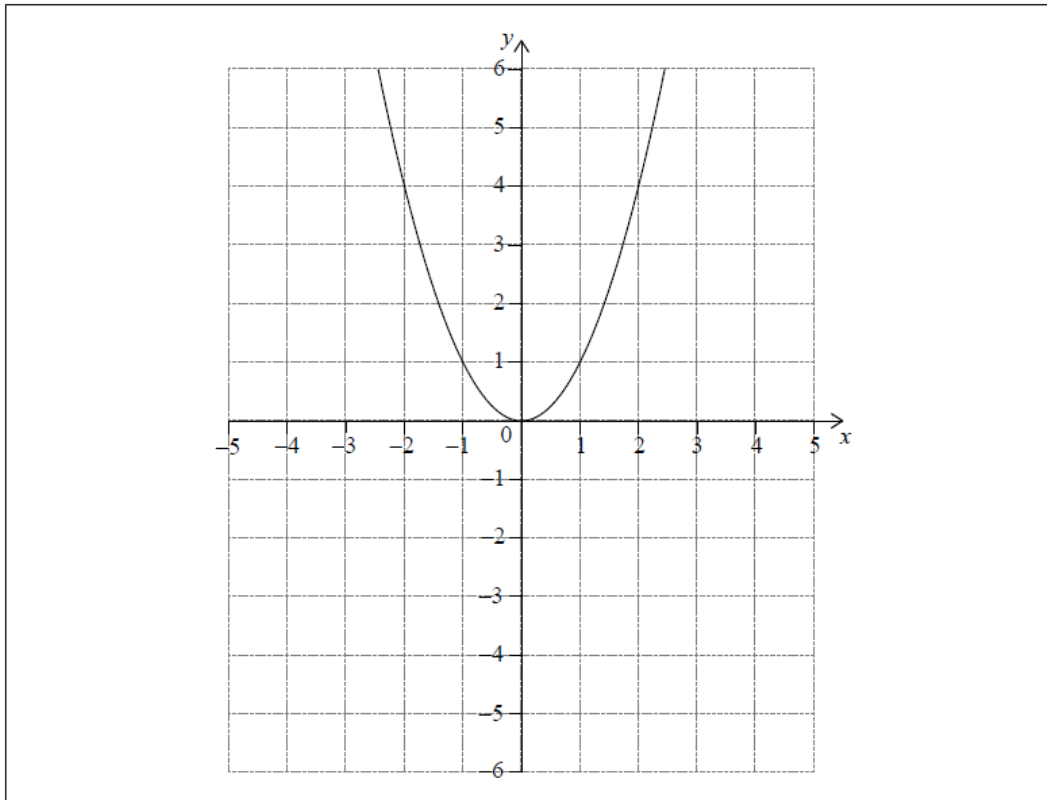
Let $g(x) = -(x-1)^2 + 5$.

(a)   Write down the coordinates of the vertex of the graph of g.   [1]

Let $f(x) = x^2$. The following diagram shows part of the graph of f.

The graph of g intersects the graph of f at $x = -1$ and $x = 2$.

(b)   On the grid above, sketch the graph of g for $-2 \le x \le 4$.   [3]

(c)   Find the area of the region enclosed by the graphs of f and g.   [3]