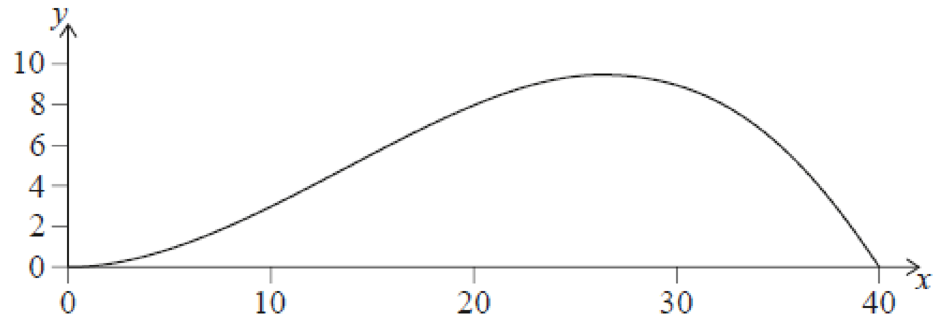
Areas (calculus) [57 marks]

1. [Maximum mark: 8] 23M.1.SL.TZ1.9

The cross section of a scale model of a hill is modelled by the following graph.



The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

(a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

Markscheme

attempt to substitute $h = 10$ and at least two different values of y into the trapezoidal rule ***(M1)***

$\frac{10}{2}\left((0+0)+2(3+8+9)\right)$

$= 200\ (\text{cm}^2)$ ***A1***

[2 marks]

It is given that the equation of the curve is
$y = 0.04x^2 - 0.001x^3,\ 0 \le x \le 40.$

(b.i) Write down an integral to find the exact cross-sectional area. [2]

Markscheme

$\int_0^{40} 0.04x^2 - 0.001x^3 \mathrm{d}x$ **OR** $\int_0^{40} y \mathrm{d}x$ ***A1A1***

Note: Award **A1** for a correct integral (including $\mathrm{d}x$), ***A1*** for correct limits in the correct location.

[2 marks]

(b.ii) Calculate the value of the cross-sectional area to two decimal places. [2]

Markscheme

$213.33\ (\mathrm{cm}^2)$ ***A2***

Note: Answer must be given to 2 decimal places to award ***A2***. Award ***A1A0*** for a correct answer given to an incorrect accuracy of at least 3 significant figures, e.g. $213\ (\mathrm{cm}^2)$.

[2 marks]

(c) Find the percentage error in the area found using the trapezoidal rule. [2]

Markscheme

attempt to substitute their parts (a) and (b)(ii) into percentage error formula ***(M1)***

$\left|\frac{213.333... - 200}{213.333...}\right| \times 100$

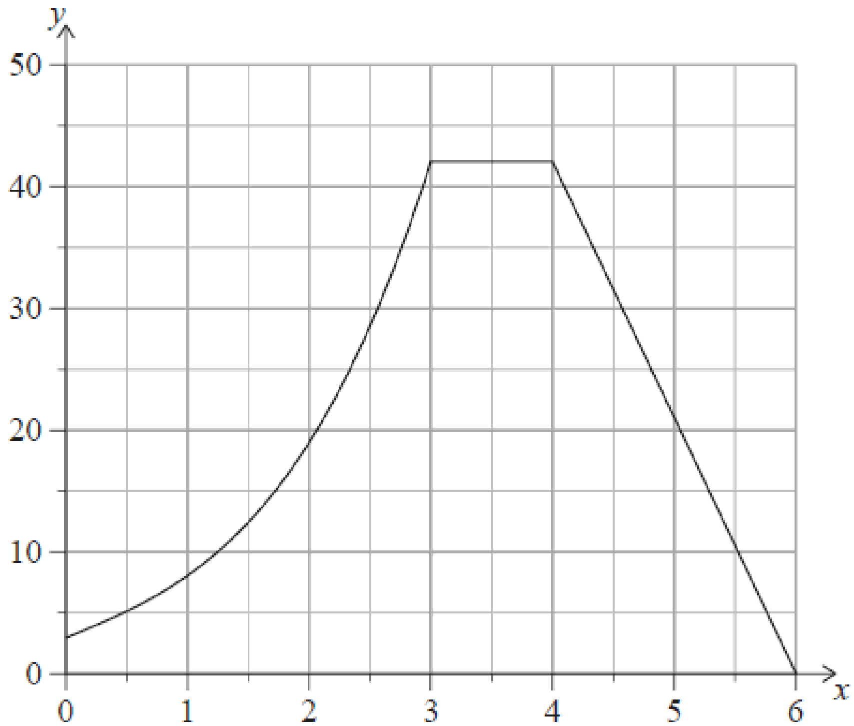
$= 6.25(\%)\ \ (6.23999...(\%))$ ***A1***

Note: Award ***(M1)A0*** for a final answer of $-6.25(\%)$ or 0.0625.

[2 marks]

2. [Maximum mark: 9] 23M.1.SL.TZ2.13

An engineer wants to calculate the cross-sectional area of a dam. The cross-section of the dam can be modelled by a curve and two straight lines as shown in the following diagram, where distances are measured in metres.



The curve is modelled by a function $f(x)$. The following table gives values of $f(x)$ for different values of x in the interval $0 \leq x \leq 3$.

x	0	0.5	1	1.5	2	2.5	3
$y = f(x)$	3	5.13	8	12.4	19	28.6	42

(a) Calculate an estimate for the area in the interval $0 \leq x \leq 3$ by using the trapezoidal rule with three equal intervals. [2]

Markscheme

attempt at using the trapezoidal rule ***(M1)***

area $= \frac{1}{2}\left(3 + 2\left(8 + 19\right) + 42\right)$

$= 49.5\ \left(\mathrm{m}^2\right)$ ***A1***

[2 marks]

It is known that $f\prime\left(x\right) = 3x^2 + 4$ in the domain $0 < x < 3$.

(b) Find an expression for $f\left(x\right)$, in the domain $0 < x < 3$. [4]

Markscheme

recognition of need to integrate (*e.g.* reverse power rule or integral symbol) ***(M1)***

$\int 3x^2 + 4\,\mathrm{d}x = x^3 + 4x + c$ ***(A1)(A1)***

Note: Award ***A1*** for each correct term.

$f\left(x\right) = x^3 + 4x + 3$ ***A1***

Note: Award ***A1*** for simplified correct answer including the value of c. Accept a value of c of 3.005 or 3.025 or 2.975 for using the non-integer x-values and their corresponding y-values.

[4 marks]

(c) **Hence** find the actual area of the **entire** cross-section. [3]

Markscheme

METHOD 1

forming expression for sum of integral and deconstructing the trapezoid into a rectangle and triangle ***(M1)***

$\int_0^3 x^3 + 4x + 3 \, \mathrm{d}x \left(= 47.25\right) + 42 \times 1 + \frac{1}{2} \times 2 \times 42 \left(= 84\right)$

(A1)

$= 131 \ \left(\mathrm{m}^2\right) \ \left(131.25\right)$ ***A1***

METHOD 2

forming expression for sum of integral and trapezoid ***(M1)***

$\int_0^3 x^3 + 4x + 3 \, \mathrm{d}x \left(= 47.25\right) + \frac{1}{2} \times 4 \times 42 \left(= 84\right)$ ***(A1)***

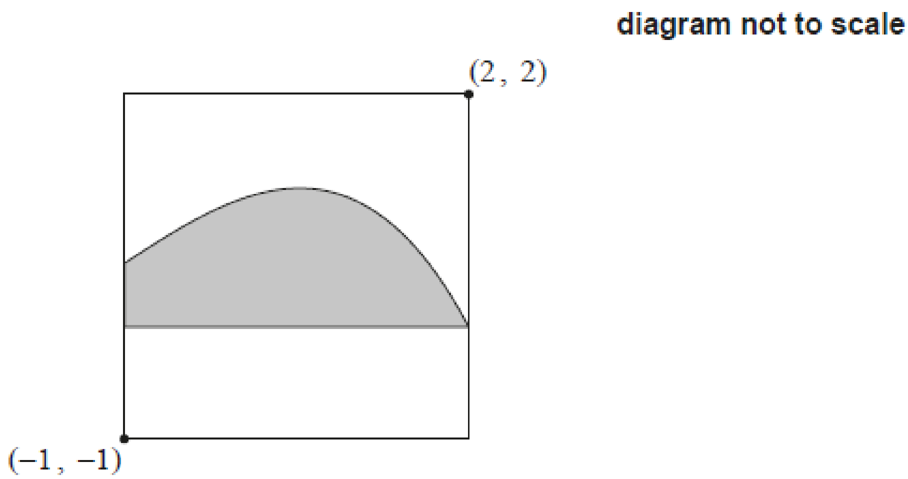
$= 131 \ \left(\mathrm{m}^2\right) \ \left(131.25\right)$ ***A1***

Note: Award ***(A1)*** for their integral with the correct limits added to 84 or their 47.25 added to 84.

[3 marks]

3. [Maximum mark: 7] 22M.1.SL.TZ1.6

A modern art painting is contained in a square frame. The painting has a shaded region bounded by a smooth curve and a horizontal line.



When the painting is placed on a coordinate axes such that the bottom left corner of the painting has coordinates $(-1,\ -1)$ and the top right corner has coordinates $(2,\ 2)$, the curve can be modelled by $y = f(x)$ and the horizontal line can be modelled by the x-axis. Distances are measured in metres.

(a) Use the trapezoidal rule, with the values given in the following table, to approximate the area of the shaded region.

x	−1	0	1	2
y	0.6	1.2	1.2	0

[3]

Markscheme

$\frac{1}{2}(0.6 + 0 + 2(1.2 + 1.2))$ ***(A1)(M1)***

Note: Award ***A1*** for evidence of $h = 1$, ***M1*** for a correct substitution into trapezoidal rule (allow for an incorrect h only). The zero can be omitted in the working.

$2.7\ \mathrm{m}^2$ ***A1***

[3 marks]

The artist used the equation $y = \frac{-x^3-3x^2+4x+12}{10}$ to draw the curve.

(b) Find the exact area of the shaded region in the painting. [2]

Markscheme

$\int_{-1}^{2} \frac{-x^3-3x^2+4x+12}{10}\, \mathrm{d}\, x$ **OR** $\int_{-1}^{2} f(x)\, \mathrm{d}\, x$ ***(M1)***

Note: Award ***M1*** for using definite integration with correct limits.

$2.925\ \mathrm{m}^2$ ***A1***

Note: Question requires exact answer, do not award final ***A1*** for 2.93.

[2 marks]

(c) Find the area of the unshaded region in the painting. [2]

Markscheme

$9 - 2.925$ ***(M1)***

Note: Award ***M1*** for 9 seen as part of a subtraction.

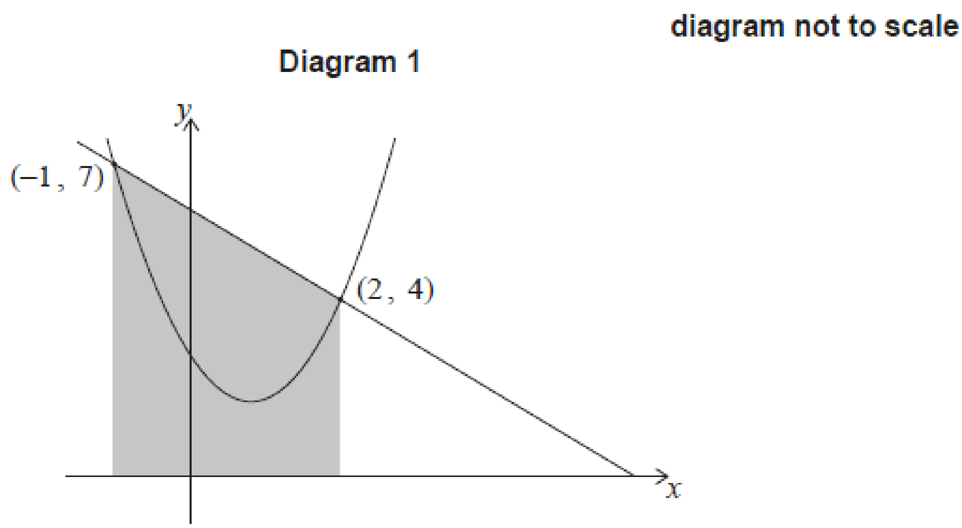
$= 6.08\ \mathrm{m}^2 \quad (6.075)$ ***A1***

[2 marks]

4. [Maximum mark: 7] 22M.1.SL.TZ2.6

The graphs of $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$ intersect at $(2,\ 4)$ and $(-1,\ 7)$, as shown in the following diagrams.

In **diagram 1**, the region enclosed by the lines $y = 6 - x$, $x = -1$, $x = 2$ and the x-axis has been shaded.



(a) Calculate the area of the shaded region in **diagram 1**. [2]

Markscheme

EITHER

attempt to substitute $3,\ 4$ and 7 into area of a trapezoid formula ***(M1)***

$(A =) \frac{1}{2}(7 + 4)(3)$

OR

given line expressed as an integral ***(M1)***

$(A =) \int_{-1}^{2}(6 - x)\ \mathrm{d}\ x$

OR

attempt to sum area of rectangle and area of triangle ***(M1)***

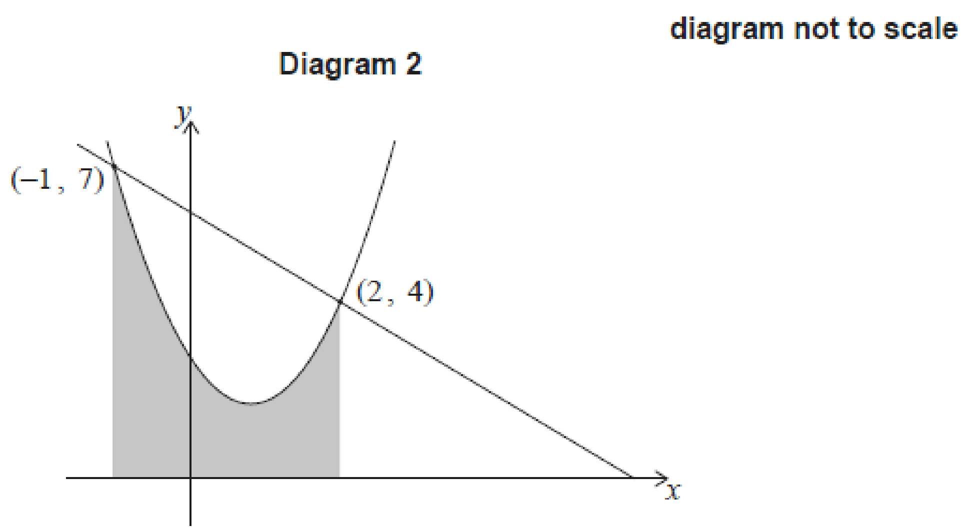
$(A =) 4 \times 3 + \frac{1}{2}(3)(3)$

THEN

16.5 (square units) ***A1***

[3 marks]

In **diagram 2**, the region enclosed by the curve $y = 1.5x^2 - 2.5x + 3$, and the lines $x = -1, x = 2$ and the x-axis has been shaded.



(b.i) Write down an integral for the area of the shaded region in **diagram 2**. [2]

Markscheme

$(A =) \int_{-1}^{2} 1.5x^2 - 2.5x + 3 \text{ d } x$ ***A1A1***

Note: Award ***A1*** for the limits $x = -1, x = 2$ in correct location. Award ***A1*** for an integral of the quadratic function, $\text{d } x$ must be included. Do not accept "y" in place of the function, given that two equations are in the question.

[2 marks]

(b.ii) Calculate the area of this region. [1]

Markscheme

9.75 (square units) ***A1***

[1 mark]

(c) Hence, determine the area enclosed between $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$. [2]

Markscheme

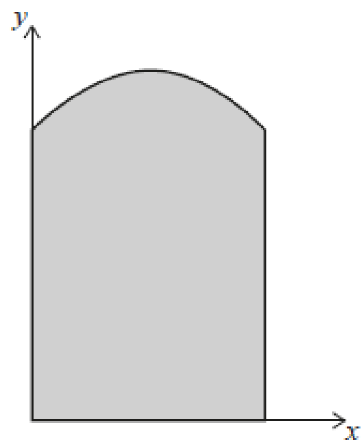
$16.5 - 9.75$ ***(M1)***

6.75 (square units) ***A1***

[2 marks]

5. [Maximum mark: 8] 21N.1.SL.TZ0.13

Irina uses a set of coordinate axes to draw her design of a window. The base of the window is on the x-axis, the upper part of the window is in the form of a quadratic curve and the sides are vertical lines, as shown on the diagram. The curve has end points $(0,\ 10)$ and $(8,\ 10)$ and its vertex is $(4,\ 12)$. Distances are measured in centimetres.



The quadratic curve can be expressed in the form $y = ax^2 + bx + c$ for $0 \leq x \leq 8$.

(a.i) Write down the value of c. [1]

Markscheme

$c = 10$ ***A1***

[1 mark]

(a.ii) Hence form two equations in terms of a and b. [2]

Markscheme

$64a + 8b + 10 = 10$ ***A1***

$16a + 4b + 10 = 12$ ***A1***

Note: Award ***A1*** for each equivalent expression or ***A1*** for the use of the axis of symmetry formula to find $4 = \frac{-b}{2a}$ or from use of derivative. Award ***A0A1*** for $64a + 8b + c = 10$ and $16a + 4b + c = 12$.

[2 marks]

(a.iii) Hence find the equation of the quadratic curve. [2]

Markscheme

$y = -\frac{1}{8}x^2 + x + 10$ ***A1A1***

Note: Award ***A1A0*** if one term is incorrect, ***A0A0*** if two or more terms are incorrect. Award at most ***A1A0*** if correct $a,\ b$ and c values are seen but answer not expressed as an equation.

[2 marks]

(b) Find the area of the shaded region in Irina's design. [3]

Markscheme

recognizing the need to integrate their expression ***(M1)***

$\int_0^8 -\frac{1}{8}x^2 + x + 10\ \mathrm{d}\ x$ ***(A1)***

Note: Award ***(A1)*** for correct integral, including limits. Condone absence of $\mathrm{d}\ x$.

$90.\,7\,\mathrm{cm}^2\ \left(\frac{272}{3},\ 90.\,6666\ldots\right)$ ***A1***

[3 marks]

6. [Maximum mark: 5] 21N.1.SL.TZ0.6

Inspectors are investigating the carbon dioxide emissions of a power plant. Let R be the rate, in tonnes per hour, at which carbon dioxide is being emitted and t be the time in hours since the inspection began.

When R is plotted against t, the total amount of carbon dioxide produced is represented by the area between the graph and the horizontal t-axis.

The rate, R, is measured over the course of two hours. The results are shown in the following table.

t	0	0.4	0.8	1.2	1.6	2
R	30	50	60	40	20	50

(a) Use the trapezoidal rule with an interval width of 0.4 to estimate the total amount of carbon dioxide emitted during these two hours. [3]

Markscheme

attempt at using trapezoidal rule formula ***(M1)***

$\frac{1}{2}\left(\frac{2-0}{5}\right)(30+50+2(50+60+40+20))$ ***(A1)***

(total carbon =) 84 tonnes ***A1***

[3 marks]

(b) The real amount of carbon dioxide emitted during these two hours was 72 tonnes.

Find the percentage error of the estimate found in part (a). [2]

Markscheme

$\left|\frac{84-72}{72}\right| \times 100\%$ ***(M1)***

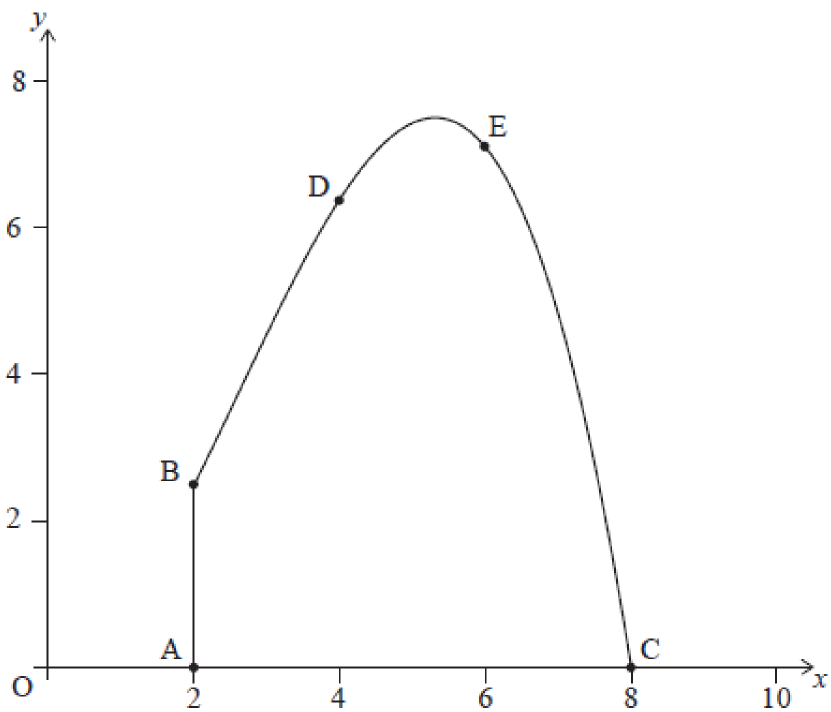
Note: Award ***(M1)*** for correct substitution of final answer in part (a) into percentage error formula.

$= 16.7\% \ \ (16.6666\ldots\%)$ ***A1***

[2 marks]

7. [Maximum mark: 13] 21M.2.SL.TZ1.5

The cross-sectional view of a tunnel is shown on the axes below. The line $[\mathrm{AB}]$ represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2,\ 2 \le x \le 8$, relative to an origin O.



Point A has coordinates $(2,\ 0)$, point B has coordinates $(2,\ 2.4)$, and point C has coordinates $(8,\ 0)$.

(a.i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

Markscheme

evidence of power rule (at least one correct term seen) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -0.3x^2 + 1.6x$ ***A1***

[2 marks]

(a.ii) Hence find the maximum height of the tunnel. [4]

Markscheme

$-0.3x^2 + 1.6x = 0$ ***M1***

$x = 5.33 \left(5.33333\ldots, \frac{16}{3}\right)$ ***A1***

$y = -0.1 \times 5.33333\ldots^3 + 0.8 \times 5.33333\ldots^2$ ***(M1)***

Note: Award ***M1*** for substituting their zero for $\frac{dy}{dx}$ $(5.333\ldots)$ into y.

$7.59\ \text{m}\ (7.58519\ldots)$ ***A1***

Note: Award ***M0A0M0A0*** for an unsupported 7.59.
Award at most ***M0A0M1A0*** if only the last two lines in the solution are seen.
Award at most ***M1A0M1A1*** if their $x = 5.33$ is not seen.

[6 marks]

When $x = 4$ the height of the tunnel is $6.4\,\text{m}$ and when $x = 6$ the height of the tunnel is $7.2\,\text{m}$. These points are shown as D and E on the diagram, respectively.

(b) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

Markscheme

$A = \frac{1}{2} \times 2((2.4 + 0) + 2(6.4 + 7.2))$ ***(A1)(M1)***

Note: Award ***A1*** for $h = 2$ seen. Award ***M1*** for correct substitution into the trapezoidal rule (the zero can be omitted in working).

$= 29.6\,\mathrm{m}^2$ ***A1***

[3 marks]

(c.i) Write down the integral which can be used to find the cross-sectional area of the tunnel. [2]

Markscheme

$A = \int_2^8 -0.1x^3 + 0.8x^2 \text{ d } x$ **OR** $A = \int_2^8 y \text{ d } x$ ***A1A1***

Note: Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if $\mathrm{d}x$ is omitted.

[2 marks]

(c.ii) Hence find the cross-sectional area of the tunnel. [2]

Markscheme

$A = 32.4 \text{ m}^2$ ***A2***

Note: As per the marking instructions, ***FT*** from their integral in part (c)(i). Award at most ***A1FTA0*** if their area is > 48, this is outside the constraints of the question (a 6×8 rectangle).

[2 marks]

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