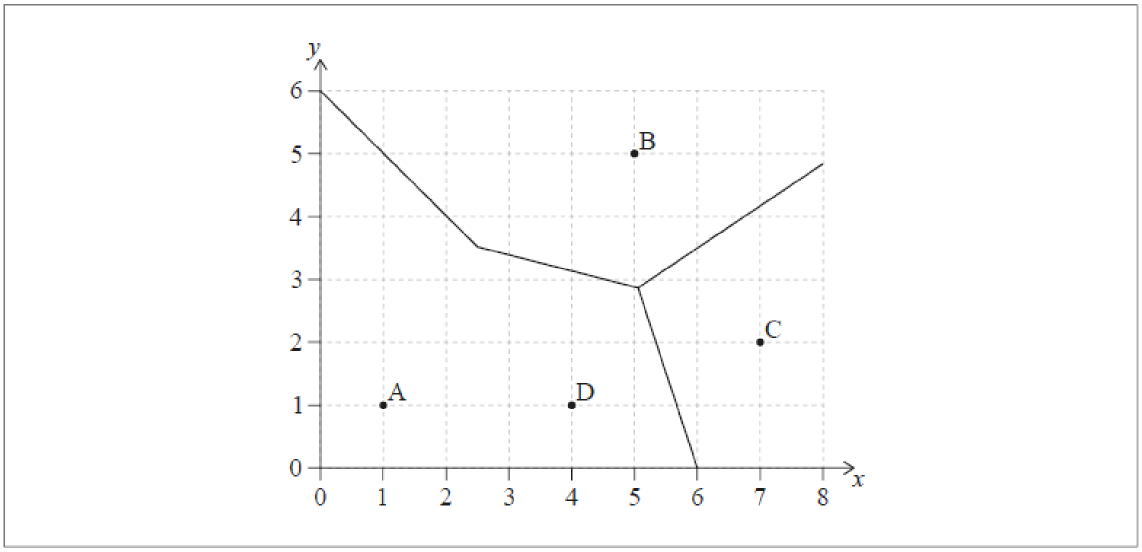
Review 3 [161 marks]

1. [Maximum mark: 6] 23M.1.SL.TZ2.7

Ani owns four cafes represented by points A, B, C and D. Ani wants to divide the area into delivery regions. This process has been started in the following incomplete Voronoi diagram, where 1 unit represents 1 kilometre.



The midpoint of CD is $(5.5,\ 1.5)$.

(a) Show that the equation of the perpendicular bisector of $[\mathrm{CD}]$ is $y=-3x+18$. [3]

Markscheme

attempt to find gradient of CD ***(M1)***

gradient of $\mathrm{CD}=\frac{1}{3}$ therefore perpendicular gradient $\mathrm{CD}=-3$ ***A1***

$y-1.5=-3(x-5.5)$ **OR** $1.5=-3(5.5)+c$ ***M1***

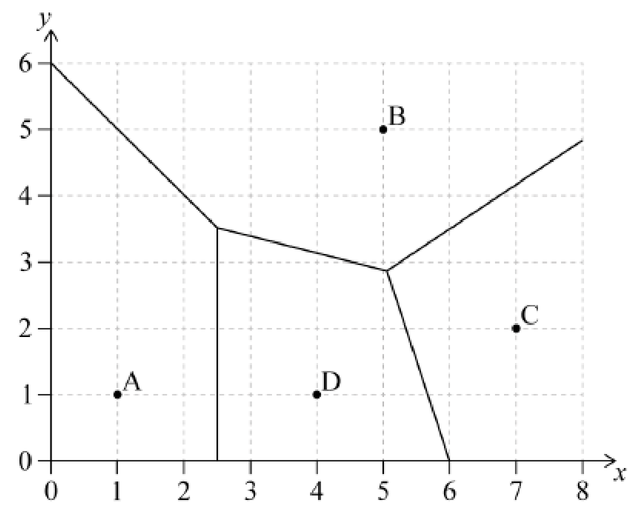
Note: Award ***M1*** for substituting the gradient and midpoint into equation of line, provided further work is seen leading to a correct answer.

$y=-3x+18$ ***AG***

[3 marks]

(b) Complete the Voronoi diagram shown above. [1]

Markscheme



perpendicular bisector AD: a vertical line with x intercept 2.5 ***A1***

Note: The perpendicular bisector should not go beyond the intersection point (should not enter site B).

[1 mark]

Ani opens an office equidistant from three of the cafes, B, C and D. The equation of the perpendicular bisector of $[\mathrm{BC}]$ is $3y = 2x - 1.5$.

(c) Find the coordinates of the office. [2]

Markscheme

attempt to solve simultaneous equations: $3y = 2x - 1.5$ and $y = -3x + 18$ ***(M1)***

$(5.05,\ 2.86)\ \ ((5.04545\ldots,\ 2.86363\ldots))$ ***A1***

Note: Accept $x = 5.05\ (5.04545\ldots)$, $y = 2.86\ (2.86363\ldots)$ in place of coordinates.

Accept $(5.05,\ 2.87)$ and $(5.05,\ 2.85)$) for using their 3 sf or 4 sf x-value to find y from any of the two equations.

[2 marks]

2. [Maximum mark: 5] 23M.1.SL.TZ2.12

In a game, balls are thrown to hit a target. The random variable X is the number of times the target is hit in five attempts. The probability distribution for X is shown in the following table.

x	0	1	2	3	4	5
$\mathbf{P}(X = x)$	0.15	0.2	k	0.16	$2k$	0.25

(a) Find the value of k. [2]

Markscheme

$0.15 + 0.2 + k + 0.16 + 2k + 0.25 = 1$ ***(M1)***

$k = 0.08$ ***A1***

[2 marks]

The player has a chance to win money based on how many times they hit the target.

The gain for the player, in $\$$, is shown in the following table, where a negative gain means that the player loses money.

x	0	1	2	3	4	5
Player's gain ($\$$)	-4	-3	-1	0	1	4

(b) Determine whether this game is fair. Justify your answer. [3]

Markscheme

$(-4 \times 0.15) + (-3 \times 0.2) + (-1 \times 0.08) + (0 \times 0.16) + (1 \times 0.16) + (4 \times 0.25)$ ***(M1)***

$= -0.12$ ***A1***

$\mathrm{E}(X) \neq 0$ therefore the game is not fair ***R1***

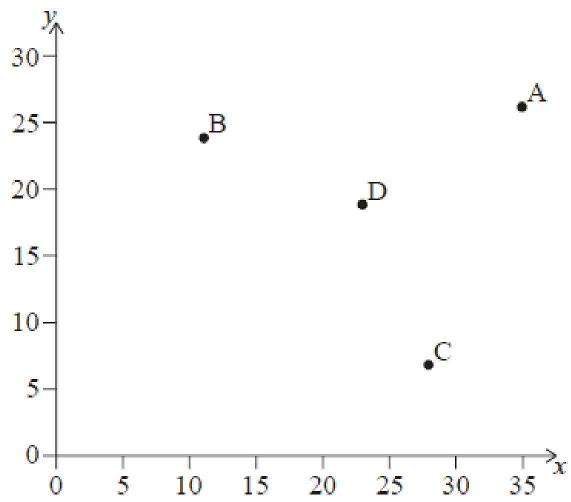
Note: Do not award ***A0R1*** without an explicit value for $\mathrm{E}(X)$ seen. The ***R1*** can be awarded for comparing **their** $\mathrm{E}(X)$ to zero provided working is shown.

[3 marks]

3. [Maximum mark: 5] 23M.1.AHL.TZ1.3

Three towns have positions $\mathrm{A}(35,\ 26)$, $\mathrm{B}(11,\ 24)$, and $\mathrm{C}(28,\ 7)$ according to the coordinate system shown where distances are measured in miles.

Dominique's farm is located at the position $\mathrm{D}(24,\ 19)$.



(a) Find AD. [2]

Markscheme

attempt to use distance formula for points D and A ***(M1)***

$$\mathrm{DA} = \sqrt{11^2 + 7^2}$$

$= 13.0$ (miles) $\left(13.0384\ldots,\ \sqrt{170}\right)$ ***A1***

Note: Accept 13 miles. Award ***M0A0*** for finding the equation of the line DA.

DA may be seen in part (b) but this should not be accepted as answer for part (a).

[2 marks]

On a particular day, the mean temperatures recorded in each of towns A, B and C are $34\ ^\circ\mathrm{C}$, $29\ ^\circ\mathrm{C}$ and $30\ ^\circ\mathrm{C}$ respectively.

(b) Use nearest neighbour interpolation to estimate the temperature at Dominique's farm on that particular day. [3]

Markscheme

$\left(\mathrm{DB} = \sqrt{13^2 + 5^2} =\right)$ 13.9 $\left(13.9283\ldots,\ \sqrt{194}\right.$ **AND**

$\left(\mathrm{DC} = \sqrt{4^2 + 12^2} =\right)$ 12.6 $\left(12.6491\ldots,\ \sqrt{160}\right.$ ***A1***

recognizing closest town is best estimate ***(M1)***

(town C is closest)

$30\,^{\circ}\mathrm{C}$ ***A1***

Note: If their DA from part (a) is the shortest length, then allow ***FT*** in (b).

[3 marks]

4. [Maximum mark: 5] 23M.1.AHL.TZ1.12

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6\ \sin\ (at + 30^\circ)$ and $V_2 = 6\ \sin\ (at + 90^\circ)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\ \sin\ (at + \theta^\circ)$.

Determine the value of V and the value of θ. [5]

Markscheme

METHOD 1 Analytical approach

attempt to express V_1 or V_2 in exponential form ***(M1)***

e.g. $V_1 = \mathrm{Im}\left(6e^{\mathrm{i}\left(at+\frac{\pi}{6}\right)}\right),\ V_2 = \mathrm{Im}\left(6e^{\mathrm{i}\left(at+\frac{\pi}{2}\right)}\right)$

Note: Accept angles in radians or degrees.

$(V_1 + V_2 =)\ 6\mathrm{e}^{\mathrm{i}\times\frac{\pi}{6}} + 6\mathrm{e}^{\mathrm{i}\times\frac{\pi}{2}}$ ***(A1)***

Note: This mark can be awarded even if seen as part of a correct larger expression.

$= 10.4\mathrm{e}^{1.05\mathrm{i}}\ \left(6\sqrt{3}\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)$ ***(A1)***

so V is $10.4\ \left(10.3923\ldots,\ 6\sqrt{3}\right)$ and θ is 60 (degrees) ***A1A1***

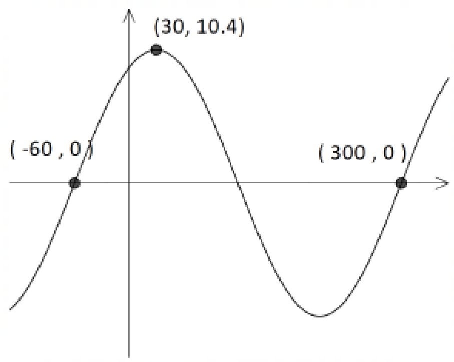
Note: Accept any value for θ that rounds to a 2sf answer of 60.

Do **not** accept a final answer for an angle in radians.

Do **not** award ***A1*** for answer of 60° resulting from incorrect working.

METHOD 2 Graphical approach

let $at = x$ and plot $V_1 + V_2$ curves on GDC ***(M1)***



attempt to find maximum *(M1)*

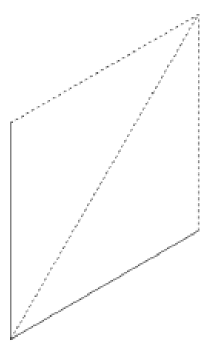
$V = 10.4$ ***A1***

attempt to find any x-axis intercept (either -60 or 300) *(M1)*

$\theta = 60$ (degrees) $\theta = -300$ (degrees) ***A1***

METHOD 3 Geometric approach

considering the rhombus



(M1)

$V = \sqrt{6^2 + 6^2 - 2 \times 6 \times 6 \cos 120^\circ}$ *(M1)*

$\left(= \sqrt{108} = 6\sqrt{3}\right) = 10.4 \ (10.3923\ldots)$ ***A1***

$\theta = 60$ (degrees) ***A2***

Note: An answer of $\theta = -300$. is most likely to be seen in METHOD 2, but should be condoned in METHODS 1 and 3 if seen there.

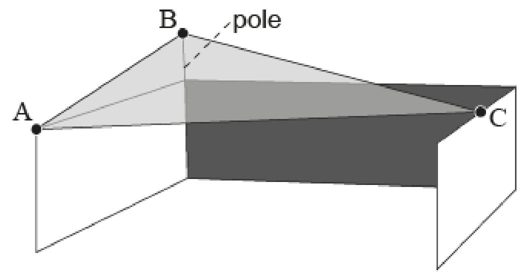
[5 marks]

5. [Maximum mark: 9] 23M.1.AHL.TZ1.7

A triangular cover is positioned over a walled garden to provide shade. It is anchored at points A and C, located at the top of a $2\,\mathrm{m}$ wall, and at a point B, located at the top of a $1\,\mathrm{m}$ vertical pole fixed to a top corner of the wall.

The three edges of the cover can be represented by the vectors

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$, $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}$, where distances are measured in metres.



(a) Calculate the vector product $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$. [2]

Markscheme

attempt to find the vector product (e.g. one term correct) ***(M1)***

$$\begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ -42 \end{pmatrix}$$ ***A1***

[2 marks]

(b) Hence find the area of the triangular cover. [2]

Markscheme

METHOD 1

attempt to use the vector product formula for the area of triangle

(condone incorrect signs and missing $\frac{1}{2}$) ***(M1)*** area $= \frac{1}{2}\sqrt{3^2 + 7^2 + 42^2}$

$= 21.3\left(\mathrm{m}^2\right)\ \left(21.3424\ldots,\ \frac{1}{2}\sqrt{1822}\right)$ ***A1***

METHOD 2

find θ using $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right| \sin\theta$ ***(M1)***

$\theta = 67.1$ ($67.1350^\circ\ldots,\ 1.171728\ldots$ radians)

then area $= \frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right| \sin\theta$

$= 21.3\left(\mathrm{m}^2\right)\ \left(21.3424\ldots,\ \frac{1}{2}\sqrt{1822}\right)$ ***A1***

[2 marks]

The point X on $[\mathrm{AC}]$ is such that $[\mathrm{BX}]$ is perpendicular to $[\mathrm{AC}]$.

(c) Use your answer to part (b) to find the distance BX. [3]

Markscheme

$AC = 7.61577\ldots\ \left(\sqrt{58}\right)$ ***(A1)***

setting the area formula $\frac{1}{2}\times$base$\times$height equal to their part (b) ***(M1)***

$\mathrm{BX} = \frac{2\times 21.3424\ldots}{\sqrt{58}}$

$= 5.60\ (5.60480\ldots)$ ***A1***

Note: Award ***A1*** for 5.6.

Award ***A1*** for $5.59\ (5.5936\ldots)$ from the use of 21.3 to 3 sf.

[3 marks]

(d) Find the angle the cover makes with the horizontal plane. [2]

Markscheme

attempting to set up a trig ratio ***(M1)***

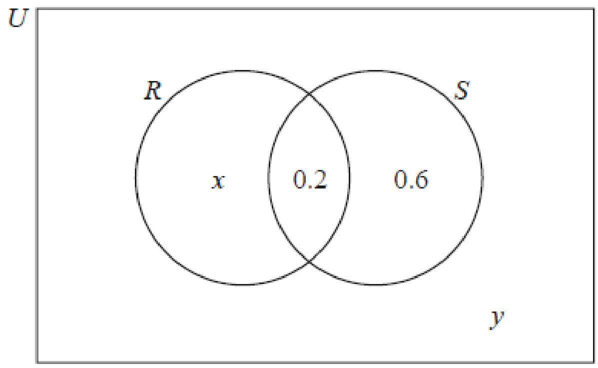
angle is $\arcsin\left(\frac{1}{\mathrm{BX}}\right)$

10.3° ($10.2776\ldots^{\circ}$, 0.179378 radians) ***A1***

[2 marks]

6. [Maximum mark: 7] 23M.1.AHL.TZ2.3

The following Venn diagram shows two independent events, R and S. The values in the diagram represent probabilities.



(a) Find the value of x. [3]

Markscheme

attempting to use $\mathrm{P}(R \cap S) = \mathrm{P}(R)\mathrm{P}(S)$ ***(M1)***

$0.2 = 0.8(0.2 + x)$ ***(A1)***

$x = 0.05$ ***A1***

[3 marks]

(b) Find the value of y. [2]

Markscheme

$x + 0.2 + 0.6 + y = 1$ ***(M1)***

$y = 0.15$ ***A1***

[2 marks]

(c) Find $\mathrm{P}(R\prime | S\prime)$. [2]

Markscheme

METHOD 1

attempting to apply $\mathrm{P}\left(R\prime\middle|S\prime\right)=\frac{\mathrm{P}(R\prime\cap S\prime)}{\mathrm{P}(S\prime)}$ ***(M1)***

$\frac{0.15}{0.2}$

$=\frac{3}{4}$ ***A1***

METHOD 2

$\mathrm{P}(R\prime|S\prime)=\mathrm{P}(R\prime)$ (because R, S are independent) ***(M1)***

$=1-0.25=0.75$ ***A1***

Note: ***FT*** from their values of x or y.

[2 marks]

7. [Maximum mark: 6] 23M.1.AHL.TZ2.7

Akar starts a new job in Australia and needs to travel daily from Wollongong to Sydney and back. He travels to work for 28 consecutive days and therefore makes 56 single journeys. Akar makes all journeys by bus.

The probability that he is successful in getting a seat on the bus for any single journey is 0.86.

(a) Determine the expected number of these 56 journeys for which Akar gets a seat on the bus. [1]

Markscheme

$(56 \times 0.86) = 48.2\ (48.16)$ ***A1***

Note: Accept 48.

[1 mark]

(b) Find the probability that Akar gets a seat on at least 50 journeys during these 28 days. [3]

Markscheme

recognizing binomial distribution (may be seen in (a)) ***(M1)***

e.g. $X \sim \mathrm{B}(56,\ 0.86)$

$(\mathrm{P}(X \geq 50) =)\ 0.316$ ***A2***

[3 marks]

The probability that Akar gets a seat on at most n journeys is at least 0.25.

(c) Find the smallest possible value of n. [2]

Markscheme

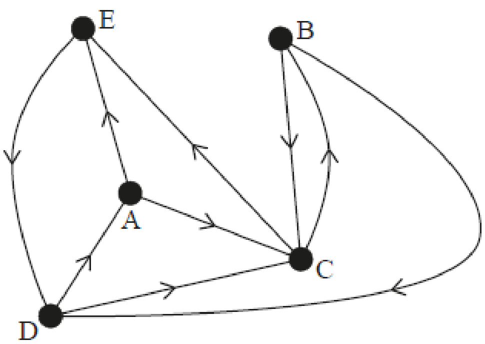
$\mathrm{P}(X \leq n) \geq 0.25$

$n = 46$ ***A2***

[2 marks]

8. [Maximum mark: 7] 23M.1.AHL.TZ2.8

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.



(a) Construct the adjacency matrix $\boldsymbol{M}$ for this network. [3]

Markscheme

attempt to create a 5x5 adjacency matrix ***(M1)***

$$\boldsymbol{M}=\begin{pmatrix}0&0&1&0&1\\0&0&1&1&0\\0&1&0&0&1\\1&0&1&0&0\\0&0&0&1&0\end{pmatrix}$$

Note: Allow the transposed matrix. Award ***A2*** for all entries correct, ***A1*** if one or two entries are incorrect, ***A0*** otherwise.

Answer presented in markscheme assumes ABCDE ordering of rows and columns; accept other orders provided they are clearly communicated.

Award ***A1*** if the zeroes are replaced by blank cells.

[3 marks]

Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

(b.i) Determine how many possible routes Beatriz could have taken, to travel from E to A. [2]

Markscheme

recognizing need to find M^7 *(M1)*

$$M^7 = \begin{pmatrix} 8 & 8 & 17 & 8 & 13 \\ 8 & 10 & 19 & 17 & 14 \\ 6 & 11 & 16 & 10 & 17 \\ 11 & 8 & 19 & 14 & 10 \\ 2 & 6 & 8 & 11 & 8 \end{pmatrix}$$

2 (routes) ***A1***

[2 marks]

(b.ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [2]

Markscheme

vertices visited in order are

EITHER

$E \to D \to C \to B \to C \to B \to D \to A$ ***A2***

OR

$E \to D \to C \to B \to C \to E \to D \to A$ ***A2***

[2 marks]

9. [Maximum mark: 9] 23M.1.AHL.TZ2.9

At a running club, Sung-Jin conducts a test to determine if there is any association between an athlete's age and their best time taken to run 100 m. Eight athletes are chosen at random, and their details are shown below.

Athlete	A	B	C	D	E	F	G	H
Age (years)	13	17	22	18	19	25	11	36
Time (seconds)	13.4	14.6	13.4	12.9	12.0	11.8	17.0	13.1

Sung-Jin decides to calculate the Spearman's rank correlation coefficient for his set of data.

(a) Complete the table of ranks.

Athlete	A	B	C	D	E	F	G	H
Age rank			3					
Time rank							1	

[2]

Markscheme

Athlete	A	B	C	D	E	F	G	H
Age rank	7	6	3	5	4	2	8	1
Time rank	3.5	2	3.5	6	7	8	1	5

A1A1

Note: Award ***A1*** for each correct row.

[2 marks]

(b) Calculate the Spearman's rank correlation coefficient, r_s. [2]

Markscheme

$r_s = -0.671\ (-0.670670\ldots)$ ***A2***

Note: Only follow through from an incorrect table provided the ranks are all between 1 and 8.

Award ***A1*** for -0.67 **OR** for the omission of the negative sign, e.g. $0.671\ (0.670670\ldots)$ or 0.67

[2 marks]

(c) Interpret this value of r_s in the context of the question. [1]

Markscheme

(A value of $r_s = -0.671$) indicates a negative correlation between a person's age and the best time they take to run $100\,\mathrm{m}$. ***R1***

Note: Condone any comment that includes "weak" or "strong" etc. Accept an interpretation in words, but only if there is a general link described and not a rule: "The older a person gets, the faster they *tend* to run". Answer must be in context.

[1 mark]

(d) Suggest a mathematical reason why Sung-Jin may have decided not to use Pearson's product-moment correlation coefficient with his data from the original table. [1]

Markscheme

Award **R1** *for any sensible reason:* ***R1***

The correlation, such that it is, is unlikely to be linear for this type of data.

Spearman's CC is less sensitive to outliers

Sung-Jin is not sure the data is drawn from a bivariate normal distribution

There are outliers/extreme data

Same time for two athletes with significantly different ages

[1 mark]

(e.i) Find the coefficient of determination for the data from the original table. [2]

Markscheme

$0.264\ (0.263762\ldots)$ ***A2***

Note: Award ***A1*** for 0.26 with no working. Given that the exact model is not specific in the question, accept correct r^2 values from other regression models: 0.631, 0.650, 0.759 and 0.256.

[2 marks]

(e.ii) Interpret this value in the context of the question. [1]

Markscheme

approximately 26% of the variability in the times taken can be explained by the runner's age. ***R1***

[1 mark]

10. [Maximum mark: 6] 23M.1.AHL.TZ2.11

Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30\ \sin\ (t + 60^\circ)$.

The voltage of the second source is modelled by the equation $V = 60\ \sin\ (t + 10^\circ)$.

(a) Determine the maximum voltage of the combined sources. [2]

Markscheme

$30\ \sin\ (t + 60^\circ) + 60\ \sin\ (t + 10^\circ)$ ***(M1)***

finding maximum graphically

$82.5\ (\mathrm{V})\ \ (82.5471\ldots)$ ***A1***

Note: Award ***M1A0*** for 83.

[2 marks]

(b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0\ \sin\ (at + b)$, where a, b and V_0 are constants, $a > 0$ and $0^\circ \leq b \leq 180^\circ$. [4]

Markscheme

recognizing that a is still 1 ***A1***

$V_0 = 82.5$ ***A1***

attempt to find an x-intercept of combined voltage ***(M1)***

$b = 26.2^\circ\ \ (26.1643\ldots^\circ)$ **OR** any other correct x-intercept ***A1***

Note: May be seen in the final answer. Award ***M1A0*** for $b = 26$ with no working.

$$(V_{\mathrm{TOT}} = 82.5\ \sin\ (t + 26.2^\circ)\ \ (82.5471\ldots\ \sin\ (t + 26.1643\ldots^\circ)))$$

Note: Award at most ***(M1)A1(A1)A0*** if phase shift of $-153.835\ldots$ is seen in the final answer. In part (b), candidates may use $\arg\left(30\mathrm{e}^{60\mathrm{i}} + 60\mathrm{e}^{10\mathrm{i}}\right)$ to determine the new phase shift, and hence could be awarded ***M1*** for this valid method.

[4 marks]

11. [Maximum mark: 7] 23M.1.AHL.TZ2.13

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, $T\prime$.

(a) Find the single matrix that represents the transformation $T\prime \rightarrow T$, which will undo the transformation described above. [4]

Markscheme

METHOD 1 (find product of matrices first)

$T \rightarrow T\prime$ is represented by $\boldsymbol{QP} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} -12 & -2 \\ 3 & 7 \end{pmatrix}$ ***(A1)***

recognizing need to find their $(\boldsymbol{QP})^{-1}$ ***(M1)***

$(\boldsymbol{QP})^{-1} = \begin{pmatrix} -12 & -2 \\ 3 & 7 \end{pmatrix}^{-1}$

$= \frac{1}{78}\begin{pmatrix} 7 & 2 \\ -3 & -12 \end{pmatrix}$ **OR** $= \begin{pmatrix} -0.0897435\ldots & -0.0256410\ldots \\ 0.0384615\ldots & 0.153846\ldots \end{pmatrix}$ ***A1***

METHOD 2 (find inverses of both matrices first)

recognizing need to find inverse of both $\boldsymbol{P}$ and $\boldsymbol{Q}$ ***(M1)***

$\boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix}$ **AND** $\boldsymbol{Q}^{-1} = \begin{pmatrix} -\frac{3}{13} & \frac{1}{13} \\ \frac{1}{13} & \frac{4}{13} \end{pmatrix}$ ***(A1)***

$T\prime \rightarrow T$ is represented by $\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ ***(M1)***

$= \frac{1}{78}\begin{pmatrix} 7 & 2 \\ -3 & -12 \end{pmatrix}$ **OR** $= \begin{pmatrix} -0.0897435\ldots & -0.0256410\ldots \\ 0.0384615\ldots & 0.153846\ldots \end{pmatrix}$ ***A1***

Note: In METHOD 1, award ***M1A0M1A0*** if they multiply the matrices in the wrong order.

In METHOD 2, award ***M1A1M1A0*** if they multiply the matrices in the wrong order.

[4 marks]

The area of $T\prime$ is $273\ \text{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

Markscheme

$\left(\det\left[-\frac{1}{78}\begin{pmatrix}7 & 2\\ -3 & -12\end{pmatrix}\right] =\right) - \frac{1}{78}$ **OR** $\left(\det\begin{pmatrix}-12 & -2\\ 3 & 7\end{pmatrix} =\right) - 78$ ***(A1)***

area of $T\prime = |\det \boldsymbol{QP}| \times \text{area of } T$ **OR** area of $T = \left|\det (\boldsymbol{QP})^{-1}\right| \times \text{area of } T\prime$ ***(M1)***

$\Rightarrow$area of $T = 273 \times \frac{1}{78}$

$= 3.5\ (\text{cm}^2)$ ***A1***

Note: Award ***(A1)(M0)A0*** for an answer of $-3.5\ (\text{cm}^2)$ with or without working. Accept an answer of $4.04\ (\text{cm}^2)$ from use of 3sf values in their answer to part (a).

[3 marks]

12. [Maximum mark: 8] 23M.1.AHL.TZ2.14

In this question, $\boldsymbol{i}$ denotes a unit vector due east, and $\boldsymbol{j}$ denotes a unit vector due north.

Two ships, A and B, are each moving with constant velocities.

The position vector of ship A, at time t hours, is given as $\boldsymbol{r}_A = (1 + 2t)\boldsymbol{i} + (3 - 3t)\boldsymbol{j}$.

The position vector of ship B, at time t hours, is given as $\boldsymbol{r}_B = (-2 + 4t)\boldsymbol{i} + (-4 + t)\boldsymbol{j}$.

(a) Find the bearing on which ship A is sailing. [3]

Markscheme

$\boldsymbol{v}_B = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ ***(A1)***

attempt to find any relevant angle ***(M1)***

$\tan^{-1}\left(\frac{3}{2}\right) \quad (= 56.3099\ldots^\circ)$

$(90^\circ + 56.3099\ldots^\circ =)\ 146^\circ \quad (146.3099\ldots^\circ)$ ***A1***

[3 marks]

(b) Find the value of t when ship B is directly south of ship A. [2]

Markscheme

setting $1 + 2t = -2 + 4t$ ***(M1)***

$t = 1.5$ (hrs.) ***A1***

[2 marks]

(c) Find the value of t when ship B is directly south-east of ship A. [3]

Markscheme

$\boldsymbol{r}_B - \boldsymbol{r}_A = (-3 + 2t)\boldsymbol{i} + (-7 + 4t)\boldsymbol{j}$ ***(M1)***

$-3 + 2t = -(-7 + 4t)$ ***(M1)***

$t = 1.67\ (\text{hrs.})\ \left(1.66666\ldots,\ \frac{5}{3}\right)$ ***A1***

[3 marks]

13. [Maximum mark: 15] 23M.2.SL.TZ1.1

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Year $^\circ\mathrm{C}$ (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708,\ 8.73)$ and $(1958,\ 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

Markscheme

$\frac{9.45-8.73}{1958-1708}$ ***(M1)***

$= 0.00288\ \left(\frac{9}{3125}\right)$ ***A1***

[2 marks]

(b.i) Interpret the meaning of the gradient in the context of the question. [1]

Markscheme

the (mean) yearly change in (mean annual) temperature ***A1***

Note: Accept equivalent statements, e.g. "rate of change of temperature".

[1 mark]

(b.ii) State appropriate units for the gradient. [1]

Markscheme

$^\circ\mathrm{C}\ /\ \text{year}$ **OR** degrees C per year ***A1***

Note: Do not follow through from part (b)(i) into (b)(ii).

[1 mark]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

Markscheme

attempt to substitute point and gradient into appropriate formula ***(M1)***

$8.73 = 0.00288 \times 1708 + c \Rightarrow c = 3.81096\ldots$

or

$9.45 = 0.00288 \times 1958 + c \Rightarrow c = 3.81096.$

equation is $y = 0.00288x + 3.81$ ***A1***

[2 marks]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Markscheme

attempt to substitute 2000 into their part (c) ***(M1)***

$0.0028 \times 2000 + 3.81096\ldots$

$= 9.57\ (^\circ\mathrm{C})\ \ (9.57096\ldots)$ ***A1***

[2 marks]

Thandizo uses linear regression to obtain a model for the data.

(e.i) Find the equation of the regression line y on x. [2]

Markscheme

$y = 0.00256x + 4.46\ \ (0.00255714\ldots x + 4.46454\ldots)$ ***(M1)A1***

Note: Award ***(M1)A0*** for answers that show the correct method, but are presented incorrectly (e.g. no "$y =$" or truncated values etc.). Accept 4.465 as the correct answer to 4 sf.

[2 marks]

(e.ii) Find the value of r, the Pearson's product-moment correlation coefficient. [1]

Markscheme

$0.861 \ \ (0.861333\ldots)$ ***A1***

[1 mark]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

Markscheme

attempt to substitute 2000 into their part (e)(i) ***(M1)***

$0.00255714 \times 2000 + 4.46454\ldots.$

$= 9.58\,(^\circ\mathrm{C})\ (9.57882\ldots(^\circ\mathrm{C}))$ ***A1***

Note: Award ***A1*** for 9.57 from $0.00255714 \times 2000 + 4.46$.

[2 marks]

Thandizo uses his regression line to predict the year when the mean annual temperature will first exceed $15\ ^\circ\mathrm{C}$.

(g) State two reasons why Thandizo's prediction may not be valid. [2]

Markscheme

cannot (always reliably) make a prediction of x from a value of y, when using a y on x line / regression line is not x on y ***A1***

extrapolation ***A1***

[2 marks]

14. [Maximum mark: 13] 23M.2.AHL.TZ1.1

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Year $^{\circ}\mathrm{C}$ (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708,\ 8.73)$ and $(1958,\ 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

Markscheme

$\frac{9.45-8.73}{1958-1708}$ ***(M1)***

$= 0.00288\ \left(\frac{9}{3125}\right)$ ***A1***

[2 marks]

(b.i) Interpret the meaning of the gradient in the context of the question. [1]

Markscheme

the (mean) yearly change in (mean annual) temperature ***A1***

Note: Accept equivalent statements, e.g. "rate of change of temperature".

[1 mark]

(b.ii) State appropriate units for the gradient. [1]

Markscheme

$^{\circ}\mathrm{C}\ /\ \mathrm{year}$ OR degrees C per year ***A1***

Note: Do not follow through from part (b)(i) into (b)(ii).

[1 mark]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

Markscheme

attempt to substitute point and gradient into appropriate formula ***(M1)***

$8.73 = 0.00288 \times 1708 + c \Rightarrow c = 3.81096\ldots$

or

$9.45 = 0.00288 \times 1958 + c \Rightarrow c = 3.81096.$

equation is $y = 0.00288x + 3.81$ ***A1***

[2 marks]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Markscheme

attempt to substitute 2000 into their part (c) ***(M1)***

$0.0028 \times 2000 + 3.81096\ldots$

$= 9.57\ (^{\circ}\mathrm{C})\ \ (9.57096\ldots)$ ***A1***

[2 marks]

Thandizo uses linear regression to obtain a model for the data.

(e.i) Find the equation of the regression line y on x. [2]

Markscheme

$y = 0.00256x + 4.46\ \ (0.00255714\ldots x + 4.46454\ldots)$ ***(M1)A1***

Note: Award ***(M1)A0*** for answers that show the correct method, but are presented incorrectly (e.g. no "$y =$" or truncated values etc.). Accept 4.465 as the correct answer to 4 sf.

[2 marks]

(e.ii) Find the value of r, the Pearson's product-moment correlation coefficient. [1]

Markscheme

$0.861 \quad (0.861333\ldots)$ ***A1***

[1 mark]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

Markscheme

attempt to substitute 2000 into their part (e)(i) ***(M1)***

$0.00255714 \times 2000 + 4.46454\ldots.$

$= 9.58\,(^\circ\mathrm{C})\ (9.57882\ldots(^\circ\mathrm{C}))$ ***A1***

Note: Award ***A1*** for 9.57 from $0.00255714 \times 2000 + 4.46$.

[2 marks]

15. [Maximum mark: 17] 23M.2.AHL.TZ1.3

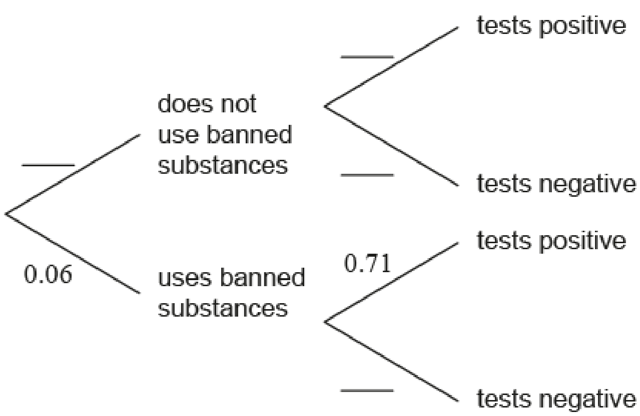
A large international sports tournament tests their athletes for banned substances.
They interpret a positive test result as meaning that the athlete uses banned substances.
A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

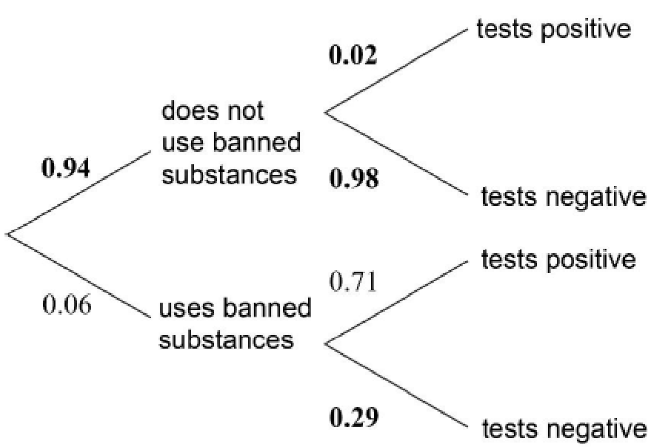
If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, complete the following tree diagram.



[2]

Markscheme



A1A1

Note: Award ***A1*** for any one value correct, ***A1*** for other three values correct. Accept percentage responses as equivalent forms on **all** branches.

[2 marks]

(b.i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative. [2]

Markscheme

multiplication of two probabilities along the tree diagram ***(M1)***

0.94×0.98

$= 0.921\ (0.9212,\ 92.1\%,\ 92.12\%)$ ***A1***

[2 marks]

(b.ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [2]

Markscheme

$(0.9212)^2$ ***(A1)***

$= 0.849\ (0.848609\ldots,\ 84.9\%,\ 84.8609\ldots\%)$ ***A1***

[2 marks]

(c.i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result. [3]

Markscheme

$0.94 \times 0.02 + 0.06 \times 0.29$ ***(A1)(M1)***

Note: Award ***A1*** for two correct products from their tree diagram seen, ***M1*** for the addition of their two products.

$0.362\ (3.62\%)$ ***A1***

[3 marks]

(c.ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [2]

Markscheme

multiplying their part (c)(i) by 1300

0.0362×1300 ***(M1)***

$47.1\ (47.06)$ ***A1***

[2 marks]

Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

Markscheme

$p = 0.02$ **OR** $p = 0.98$ ***(A1)***

recognition of binomial probability with $n = 20$ ***(M1)***

$\mathrm{P}(X = 0)$ **OR** $\mathrm{P}(X = 20)$ ***(M1)***

$0.668\ (0.667607\ldots)$ ***A1***

Note: Award ***(A1)(M1)(M1)A0*** for an answer of 0.667.

$0.98^{20} = 0.668\ \left(0.667607\ldots\right)$ is awarded full marks.

[4 marks]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

Markscheme

$\mathrm{P}(X \geq 3)$ **OR** $\mathrm{P}(X \leq 17)$ ***(M1)***

$0.00707\ (0.00706869\ldots)$ ***A1***

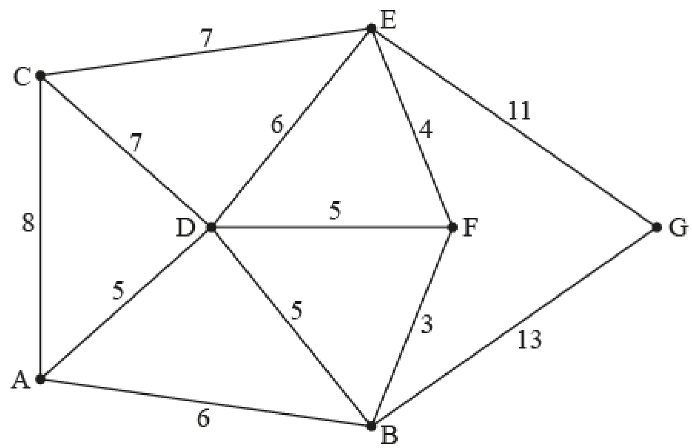
Note: Award ***(M1)A0*** for an answer of 0.00706. Award ***(M1)A0*** for an answer of 0.0599 $(0.0598989\ldots)$, obtained from the use of $\mathrm{P}(X \geq 2)$.

FT from their value of p in part (d)

[2 marks]

16. [Maximum mark: 17] 23M.2.AHL.TZ1.4

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

Markscheme

there are more than two vertices with odd degree ***R1***

so it is not possible to travel along each road exactly once ***A1***

Note: Do not award ***R0A1***.

Award ***R1*** for "There are 4 vertices with odd degree".

[2 marks]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	***a***	***b***
D	5	5	7		6	5	***c***
E	11	7	7	6		4	11
F	9	3	***a***	5	4		***d***
G	19	13	***b***	***c***	11	***d***	

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

Markscheme

$a = 11, b = 18, c = 17, d = 15$ ***A2***

Note: Award ***A1*** for any one correct, ***A2*** for all four correct.

[2 marks]

(c) Use the nearest neighbour algorithm, starting at vertex G to find an upper bound for the travelling salesman problem. [3]

Markscheme

attempt to use nearest neighbour algorithm ***(M1)***

Note: Award ***M1*** for first 3 vertices correct or $11, 4, 3$ seen.

$\mathrm{G–E–F–B–D–A–C(–E)–G}$ **OR** $11 + 4 + 3 + 5 + 5 + 8 +$ their b ***(A1)***

upper bound = $54\,(\mathrm{km})$ ***A1***

[3 marks]

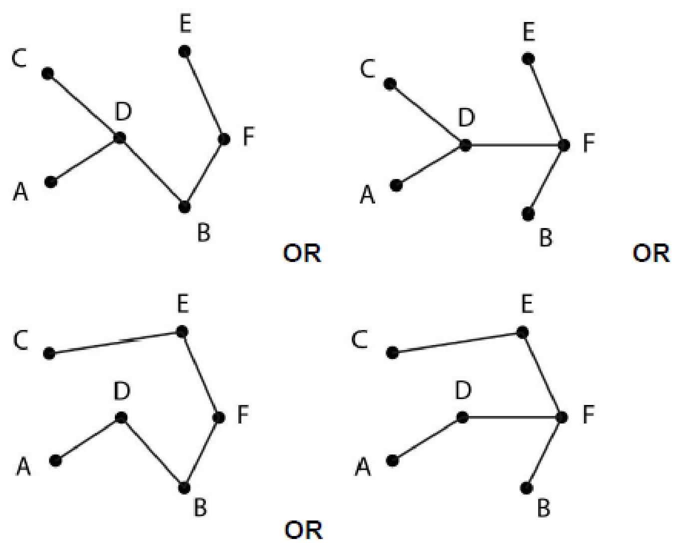
(d.i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F}$. [2]

Markscheme

a diagram of **any** spanning tree of the subgraph ABCDEF ***(A1)***

attempt at Kruskal's algorithm or Prim's algorithm ***(M1)***

e.g. edges BF (3), EF (4) and an edge of length 5 listed or seen in any spanning tree



A1

(d.ii) Write down the total weight of the minimum spanning tree. [2]

Markscheme

$24\,(\mathrm{km})$ ***A1***

Note: ***FT*** from their sketch, only if it is a spanning tree. It is not required to see the edge lengths on the sketch, since they are given in the question.

[2 marks]

(e) Hence find a lower bound for the travelling salesman problem. [2]

Markscheme

adding vertex G's two shortest edges to their part (d)(ii) ***(M1)***

$24 + 11 + 13$

$= 48$ ***A1***

[2 marks]

(f) Explain one way in which an improved lower bound could be found. [1]

Markscheme

try removing a different vertex ***A1***

[1 mark]

It is found that the optimum solution starting at A is actually $\mathrm{A–C–E–G–B–F–D–A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

Markscheme

recognize 7 edges in optimum route ***(M1)***

Note: Award ***M1*** for a total length of 52 seen.

subtracting $0.5\times$ edges from 52 ***(M1)***

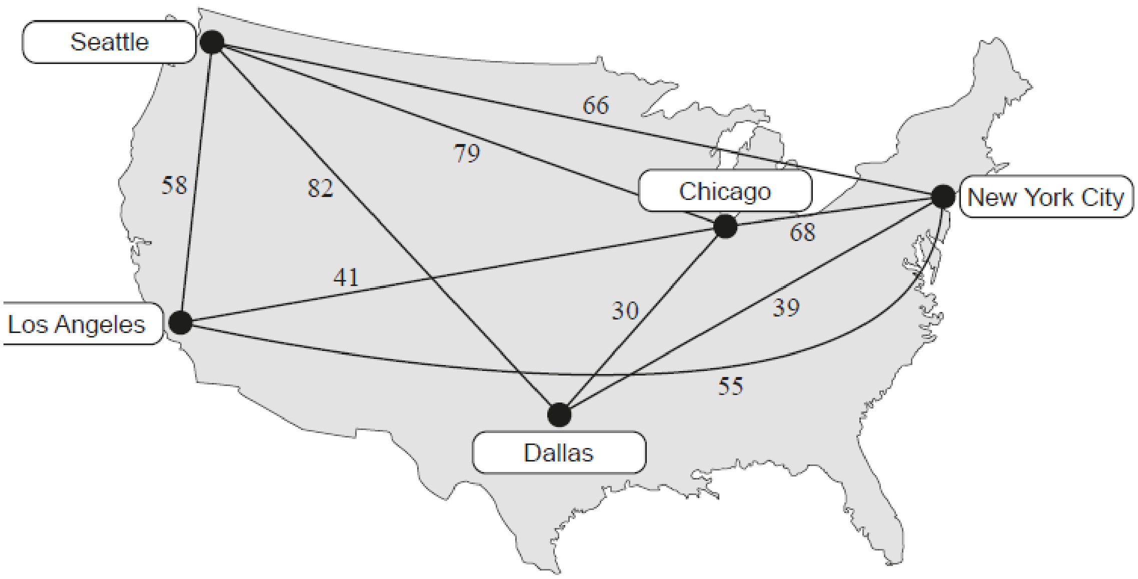
$52 - 7 \times 0.5$

$= 48.5\,(\mathrm{km})$ ***A1***

[3 marks]

17. [Maximum mark: 19] 23M.2.AHL.TZ2.4

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($\$$) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

Markscheme

any city can be travelled to or from any other city (so is connected) ***R1***

EITHER

but there is no direct flight between Los Angeles and Dallas (for example) ***R1***

OR

but not every vertex has degree 4 ***R1***

Note: Accept equivalent statements for the cities being connected and the graph not being complete.

[2 marks]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

Markscheme

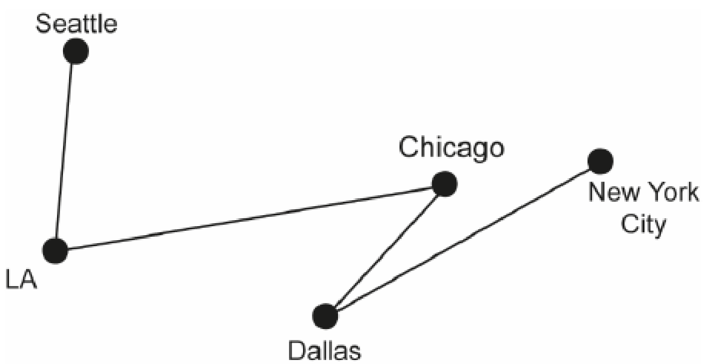
edge CD selected first ***M1***

DN,

CL,

LS ***A1***

Note: Award marks if the answers are written as sums in the correct order. ***M1*** if 30 is seen first, ***A1*** for $30 + 39 + 41 + 58$.



A1

Note: The final ***A1*** can be awarded independently. Award ***M0A0A1*** for a correct MST graph with no other working. Award ***M1A0A1*** if Prim's algorithm is seen to be used correctly with CD first.

[3 marks]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Markscheme

$2 \times \mathrm{MST}$ weight ***(M1)***

$= \$336$ ***A1***

Note: Allow any integer multiple (> 1) of MST weight for ***M1,*** and if correctly calculated, award ***M1A1.***

[2 marks]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

Markscheme

attempt at nearest neighbour algorithm ***M1***

order is $\text{LA} \rightarrow \text{D} \rightarrow \text{C} \rightarrow \text{NYC} \rightarrow \text{S} \rightarrow \text{LA}$ ***A1***

Note: Award ***M1*** for a route that begins with LA and then D, this includes seeing 26 as the first value in a sum. Award ***A1*** if $26 + 30 + 68 + 66 + 58$ seen in order.

Note: Award ***M1A0*** for an incorrect first nearest neighbour proceeding 'correctly' to the next vertex. For example, LA to C and then C to D.

upper bound is $(26 + 30 + 68 + 66 + 58 =)\ \$248$ ***A1***

Note: Award ***M1A0*** for correct nearest neighbour algorithm starting from a vertex other than LA. Condone the correct tour written backwards i.e. $58 + 66 + 68 + 30 + 26 = 248$

[3 marks]

(e.i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem. [3]

Markscheme

attempt to find MST of L, N, D and S ***(M1)***

by deleting C, Kruskal gives MST for the remainder as LD, DN, LS weight 123 ***(A1)***

(lower bound is therefore $123 + (30 + 41) =)\$194$ ***A1***

Note: Award ***(M1)*** for a graph or list of edges that does not include C.

Award ***(A1)*** if $26+39+58$ seen in any order.

[3 marks]

(e.ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [2]

Markscheme

by deleting S, Kruskal gives MST for the remainder as LD, DC, DN weight 95 ***(A1)***

(lower bound is therefore $95+(58+66)=)\ \$219$ ***A1***

Note: Award ***(A1)*** if $26+30+39$ seen in any order.

[2 marks]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

Markscheme

$219 \leq C \leq 248$ ***A1A1***

Note: Award ***A1*** for $219 \leq C$ and ***A1*** for $C \leq 248$. Award at most ***A1A0*** for $219 < C < 248$. ***FT*** for their values from part (e) if higher value from (e)(i) and (e)(ii) used for the lower bound, and part (d) for the upper.

[2 marks]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

Markscheme

any valid tour, within their interval from part (f), from any starting point **OR**
any valid tour that starts and finishes at N ***(M1)***

valid tour starting point N **AND** within their interval ***A1***
e.g. NDCLSN (weight 234)

Note: If part (f) not correct, only award ***A1FT*** if their valid tour begins and ends at N **AND** lies within **BOTH** their interval (including if one-sided) in part (f) **AND** $219 \leq C \leq 248$.

If no response in the form of an interval seen in part (f) then award ***M1A0*** for a valid tour beginning and ending at N **AND** within $219 \leq C \leq 248$.

[2 marks]